CONDUCTION HEAT TRANSFER

by

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PREFACE

This book is written for engineering students and for engineers working in heat transfer research or on thermal design. A new book on conduction is going to give rise to a number of questions among the members of the latter group: What conceivable reason inspired the author to write another book on conduction? Do we not already have the first and last testaments of conduction by Fourier and by Carslaw and Jaeger? What is the importance of conduction in today's engineering heat transfer studies? And so on. After all, is it not the feeling among engineers that the temperature problems associated with solids are now classical, with many solutions existing in the literature for a number of geometries and boundary conditions? Doubtless, from the viewpoint of mathematics, the foregoing points are all quite valid. One cannot claim, however, that the foundations of engineering conduction are based on mathematics only. And this text is intended, not to serve as an additional catalog of a number of new situations which are not listed in Carslaw and Jaeger, but rather to introduce the reader to engineering conduction.

Problems of engineering heat transfer involve one or a combination of the phenomena called *diffusion, radiation, stability,* and *turbulence.* Among these phenomena diffusion, because of its comparative simplicity, is a logical starting point in the study of heat transfer. In other words, we are not interested in diffusion for its own sake, as was the case for Fourier and for Carslaw and Jaeger. For this reason, the traditional mathematical treatment of the subject is no longer adequate. In order to provide an exposition of applied nature, I have followed the philosophy which I thought would be most suitable to engineering—the combination of physical reasoning with theoretical analysis. I regret that even in a book of this size, I have not been able to do justice to another but equally important aspect of heat transfer, experimental methods. Instead of defending the content of the text, however, I prefer simply to admit that I have chosen to write about only those topics in which I have some confidence of my understanding.

In the planning stage, my intention was to write a text involving the linear diffusion of momentum, mass, electric current, and neutrons as well as heat. This, of course, is a more elegant way to demonstrate the phenomenon of diffusion. Yet, without sacrificing a part of the present text and considerably exceeding the size of a typical text book, there seems to be no way of accomplishing this task. I thus confined myself to writing on the diffusion of heat only. The diffusion of heat in a rigid medium differs from that in a deformable medium, the latter including the diffusion of momentum. From the standpoint of the formulation, the rigid medium is a special case of the deformable medium, and

need not be considered separately. From the viewpoint of solutions, however, the techniques applicable to a deformable body are in general not convenient for the linear problems of a rigid body, because of the nonlinear nature of the equations which govern the diffusion of momentum. For this reason, I have devoted the text to diffusion of heat in regid media, the so-called conduction phenomenon only.

Following an introductory chapter, the text is divided into three parts. In Part I, *Formulation,* I have tried hard to break away from the traditional thought that the formulation of conduction problems is merely

$$\frac{dT}{dt} = \nabla \cdot (k \nabla T) + \frac{u'''}{\rho c},$$

or another but similar differential equation. I have kept this part somewhat general so it can readily be extended to the case of deformable media. Only the treatment of inertial coordinates, stress tensor, momentum, and moment of momentum are omitted from this discussion. I have devoted Part II, *Solution,* to the simplest and, to a large extent, the general (but not necessarily the most elegant) methods of solution. Thus the potential theory, the source theory, Green's functions, and the transform calculus (with the exception of Laplace transforms) are left untreated. This seems quite adequate for the intended size and level of the text. I have collected topics of advanced or special nature under Part III as *Further Methods of Formulation and Solution.* These include variational calculus, difference and differential-difference formulations, and relaxation, numerical, graphical, and analog solutions.

In general, problems are designed to clarify the physically and/or mathematically important points, and to supplement and extend the text. The classical "handbook" type of material is avoided as much as possible. Aside from the classical problems, the great majority are my own inventions.

With few exceptions, no more engineering background is required of the reader than the customary undergraduate courses in thermodynamics, heat transfer, and advanced calculus. Prior knowledge in fluid mechanics and the fundamentals of vector calculus are helpful but not necessary. The sections involving vectors may be studied without them by using a coordinate system in the usual manner.

The text is the result of a series of revisions of the material originally prepared and mimeographed for use in a senior-graduate course on heat transfer in the Mechanical Engineering Department of The University of Michigan. It may, of course, find application in other fields of endeavor which deal with temperature and associated stress problems in solids.

The following outline and suggestions seem pertinent for a three-credit course. Chapter 1 should be read as a survey based on undergraduate material. Other examples than those employed in this chapter may be utilized, the choice depending on the instructor's taste and the students' background. Chapter 2 is the most important chapter of the text. It may not be possible to master this chapter in one attempt. It is therefore strongly suggested that the chapter be continuously reviewed in the course of study. Elementary parts of Chapter 3 may be eliminated for students who have had a first course on heat transfer. Chapters 4, 5, and 6 are the backbone of solution methods, and should be studied without omission. In general, the time available for a three-credit course does not permit study of all the remaining chapters. For the rest of the course, therefore, one or possibly two chapters out of Chapters 7, 8, 9, and 10 are suggested; again, the choice will depend on the instructor's taste and the students' background.

My first acknowledgment is to my student, friend, and now colleague Professor P. S. Larsen, who read my notes in the course of the writing process and made invaluable suggestions. Thanks are extended to Professor R. J. Schoenhals of Purdue University and Professor J. W. Mitchell of the University of Wisconsin for their constructive criticism of the manuscript. I am grateful to Professor G. J. Van Wylen, then Chairman of the Mechanical Engineering Department and now Dean of Engineering, and to Professors W. Mirsky, H. Merte, and J. R. Cairns of The University of Michigan for reading one chapter of my notes and making some remarks. I am also indebted to Professor G. J. Van Wylen for reducing my teaching load for one semester in the final stage of my writing. Professor J. A. Clark, Professor-in-charge of the Heat Transfer Laboratory, has been a continual source of encouragement and inspiration as a friend and colleague. I am thankful to my students, to Professor C. L. S. Farn of Carnegie Institute of Technology, and to Messrs. L. H. Blake and C. Y. Warner, for helping me in the preparation of some figures.

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June 1966

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The ability to analyze a problem involves a combination of inherent insight and experience. *The former, unfortunately, cannot be learned, but depends on the individual. However, the latter is of equal importance, and can be gained with* patient study.

INTRODUCTION

FOUNDATIONS OF HEAT TRANSFER

The foundations of any engineering science may best be understood by considering the place of that science in relation to other engineering sciences. Therefore, our first concern in this chapter will be to determine the place of heat transfer among the engineering sciences. Next, two modes of heat transfer —diffusion and radiation—will be briefly reviewed. We shall then proceed to a discussion of the *continuum* and the *molecular* approaches to engineering problems, and finally, to a discussion of the foundations of *continuum heat transfer.*

1–1. The Place of Heat Transfer in Engineering

Let us first review four well-known problems taken from the mechanics of rigid and deformable bodies and from thermodynamics. For each problem let us consider two formulations, based on different assumptions. Our concern will be with the nature of the physical laws employed in these formulations. (At this stage, our discussion will necessarily be framed in the conventional terms of existing textbooks; the philosophy of the present text will be set forth in the next chapter.)

Example 1–1. *Free fall of a body.* Consider a body of mass m falling freely under the effect of the gravitational field (Fig. 1–1). We wish to determine the instantaneous location of this body.

Formulation (physics) of the problem: Newton's second law of motion,

$$\mathbf{F} = m\mathbf{a}, \tag{1–1}$$

F being the sum of external forces and **a** the acceleration vector, may be reduced to a one-dimensional problem, for if we neglect the resistance R of the surrounding medium, Eq. (1–1) can be written in the form

$$mg = m\frac{d^2x}{dt^2}, \tag{1–2}$$

subject to the appropriate initial conditions.

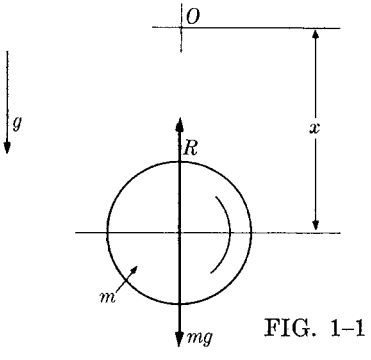


FIG. 1–1

Solution (mathematics) of the problem: Integrating Eq. (1–2) twice with respect to time yields

$$x = \tfrac{1}{2}gt^2 + C_1t + C_2, \tag{1–3}$$

where the two integration constants, C_1 and C_2, may be determined from the initial position and velocity of the body.

The formulation and solution of a problem are clearly distinguished in the foregoing trivial case, elaborated for this purpose.

In our second formulation of the problem, let us include the resistance to the motion of the body exerted by the ambient medium. With this consideration we have

$$mg - R = m\frac{d^2x}{dt^2}. \tag{1–4}$$

Equation (1–4) cannot be integrated without further information about the resistance force R. If, for example, this force is assumed to be proportional to the square of the velocity of the body, that is, if

$$\frac{R}{m} = k\left(\frac{dx}{dt}\right)^2, \tag{1–5}$$

then Eq. (1–4) gives

$$g - k\left(\frac{dx}{dt}\right)^2 = \frac{d^2x}{dt^2}, \tag{1–6}$$

where k is a constant. Equation (1–6) is a nonlinear differential equation whose solution is quite involved, and since this solution is unimportant for the present discussion it will not be given here.

Before commenting on the foregoing formulations, let us consider two more problems taken from mechanics.

Example 1–2. *Reaction forces of a beam.* Consider a beam subject to a localized force P. We wish to determine the reaction forces of the beam.

For the first formulation of the problem let us assume that the beam is simply supported (Fig. 1–2). The reaction forces A and B may then be obtained from the conditions

$$\sum \text{Force} = 0, \qquad \text{Force balance,} \tag{1–7}$$

$$\sum \text{Moment} = 0, \qquad \text{Moment balance,} \tag{1–8}$$

which are the results of Newton's second law of motion as applied to statics problems.

For the second formulation of the problem let us replace one of the simple supports of the beam by a built-in support as shown in Fig. 1–3. The new case can no longer be solved by employing Newton's law only. Because there are three unknowns, the reaction forces A_1, B_1, and the bending moment M, we require one more condition in addition to Eqs. (1–7) and (1–8). This may be obtained by considering the nature of the beam. If, for example, the beam is assumed to be elastic, the additional condition may be derived from *Hooke's law.*

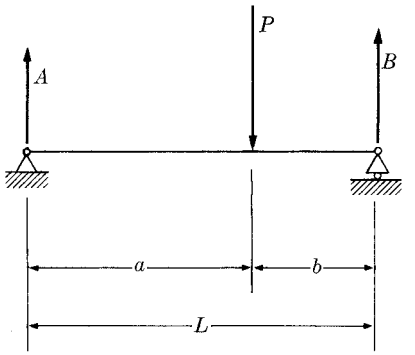


FIG. 1–2

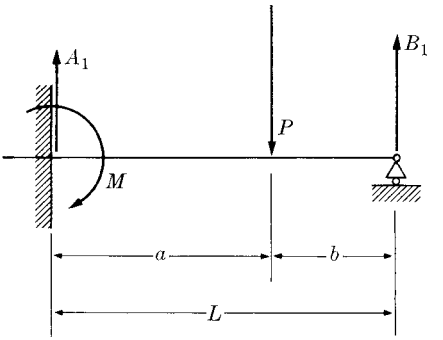


FIG. 1-3

As in Example 1–1, the details of the foregoing formulations and their solutions are not important for the present discussion.

Example 1–3. *Fully developed, steady incompressible flow of a fluid between two parallel plates.* We wish to find the velocity distribution in the flow.

Consider the control volume* shown in Fig. 1–4. Since the acceleration terms are identically zero, when Newton's second law of motion is applied to this control volume it again reduces to a force balance. The normal and tangential components of the *surface forces* per unit area are designated pressure p and shear τ. The *body forces* such as gravity are neglected in this example. The first formulation of the problem will be based on the assumption of an ideal (frictionless) fluid, the second on that of a viscous (Newtonian) fluid.

If the fluid is ideal, the shear stress is zero by definition, and the pressure change in the x-direction is also zero as a consequence of Newton's second law. The fluid may now be replaced by infinitely thin parallel layers with no friction between them. In the absence of any net force, *Newton's first law* states that each layer must either be stationary or move with a uniform velocity. Therefore, if the entrance velocity is constant and uniform, this condition is preserved axially and transversally for all values of the time. In other words, the velocity at any cross section is uniform.

If fluid friction is included, Newton's second law applied to the control volume of Fig. 1–4 yields

$$-\frac{d\tau}{dy} - \frac{dp}{dx} = 0. \tag{1–9}$$

Equation (1–9), however, cannot be solved to give the desired velocity distribution unless an additional condition is specified. One such condition might be a relation between the shear stress and the velocity gradient. If we assume,

* The definition of *control volume* will be given in the next chapter.

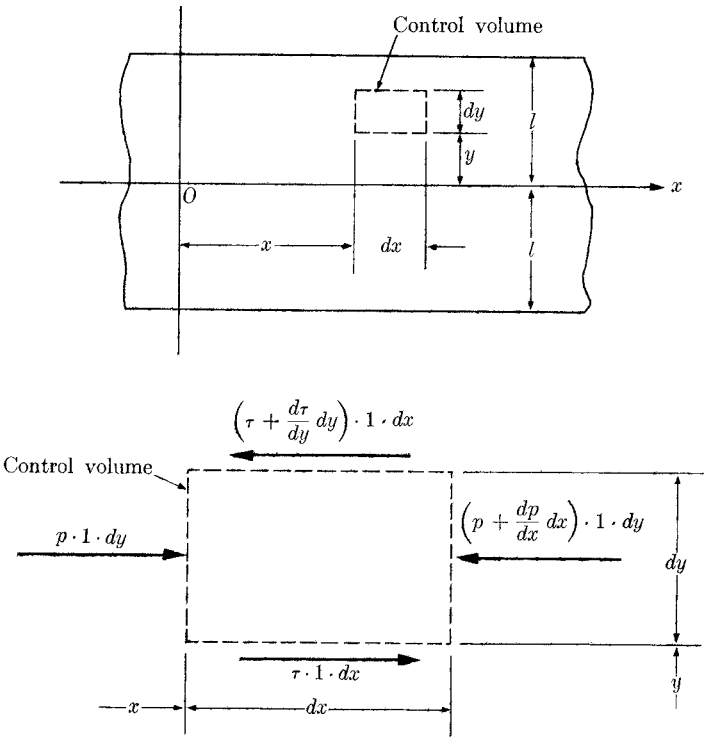


FIG. 1-4

for example, that the fluid is Newtonian, this condition may be stated as

$$\tau = -\mu \frac{du}{dy}, \tag{1–10}$$

where the proportionality constant is the viscosity of the fluid. Introducing Eq. (1–10) into Eq. (1–9) and rearranging gives

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\left(\frac{dp}{dx}\right). \tag{1–11}$$

Equation (1–11) and the boundary conditions

$$\frac{du(0)}{dy} = 0, \qquad u(l) = 0, \tag{1–12}$$

complete the formulation of the problem. For a constant axial pressure gradient (dp/dx), the solution of Eq. (1–11) satisfying Eq. (1–12) is the well-known parabolic velocity distribution

$$u(y) = \frac{(-dp/dx)l^2}{2\mu}\left[1 - \left(\frac{y}{l}\right)^2\right]. \tag{1–13}$$

Let us now examine the foregoing three examples in the light of the physical laws used in their formulations. As we have just seen, some problems taken from mechanics can be solved by using only Newton's second law of motion, combined sometimes with Newton's first law and/or the conservation of mass; these are called *mechanically determined problems.* The dynamics of rigid bodies in the absence of friction, the statically determined problems of rigid bodies, and the mechanics of ideal fluids provide well-known examples of this class. More specifically, we may place in this category the first formulations of each of the foregoing problems: the free fall of a body without friction, the statically determined beam, and the steady flow of an ideal fluid between two parallel plates.

Some mechanics problems, however, require an extra condition in addition to Newton's laws of motion and the conservation of mass. These are called *mechanically undetermined problems.* The dynamics of rigid bodies with friction and the mechanics of deformable bodies (viscous, elastic, plastic, viscoelastic bodies) provide examples of this group, as illustrated by the second formulations of each of the foregoing examples: the free fall of a body with friction, the statically undetermined beam, and laminar flow between two parallel plates. It is important to note that each of these mechanically undetermined problems employs not only the general laws of mechanics, but also an additional law whose nature depends on the specific problem under consideration. The free fall of a body requires a relation between the resistance force and the velocity, the statically undetermined beam requires a relation between the stress and the strain, and laminar flow between two parallel plates requires a relation between the shear stress and the velocity. Hereafter any such additional law will be called a *particular law,* although the term *constitutive relation* is more frequently used in the literature.

Thermodynamics problems may similarly be divided into two classes. Some thermodynamics problems can be solved by employing the *general* (first and second) *laws of thermodynamics* and, if necessary, the general laws of mechanics; these are called *thermodynamically determined problems.* Others, however, require the use of conditions in addition to the general laws; these are called *thermodynamically undetermined problems.* The following example may be helpful to illustrate the point.

Example 1–4. *Steady one-dimensional isentropic and subsonic flow of an inviscous fluid through an insulated diffuser.* The state of the fluid is given at the inlet; we wish to find the state at the outlet. The notation is shown in Fig. 1–5—the pressure p, the density ρ, the internal energy u, the

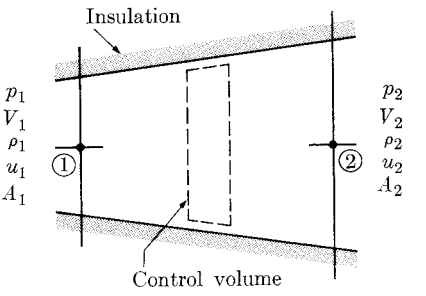


FIG. 1–5

velocity V, and the cross-sectional area A. The inlet properties are identified by the subscript 1 and the outlet properties by the subscript 2.

Let us apply the appropriate general laws to the control volume shown in Fig. 1–5. The law of *conservation of mass* gives

$$d(\rho A V) = 0; \tag{1–14}$$

Newton's second law of motion, rearranged by means of Eq. (1–14), results in

$$dp + \rho V\, dV = 0, \tag{1–15}$$

and the *first law of thermodynamics*, combined with Eqs. (1–14) and (1–15), yields

$$du + p\, d(1/\rho) = 0. \tag{1–16}$$

In the first formulation of the problem we assume that the fluid is incompressible. Then by definition, $\rho_1 = \rho_2$, and the remaining outlet properties V_2, p_2, u_2 are obtained from the integration of Eqs. (1–14), (1–15), and (1–16), respectively, between the inlet and the outlet of the diffuser.

In the second formulation of the problem we consider instead a compressible fluid. Now, in addition to V_2, p_2, and u_2, the outlet density ρ_2 must be determined; therefore, the three conditions given by Eqs. (1–14), (1–15), and (1–16) are no longer sufficient. To complete the formulation, let us recall the manner in which we arrived at the second formulations of Examples 1–1, 1–2, and 1–3. In the present example, the required additional condition may be related to the nature of the fluid. Implicitly, we may write this condition in the form

$$p = p(\rho, u). \tag{1–17}$$

In practice, Eq. (1–17) is expressed or tabulated in a number of ways suitable for frequently encountered problems. The most commonly found explicit form of Eq. (1–17) is

$$p = \rho RT, \tag{1–18}$$

the so-called *ideal gas law*. Equations (1–17) and (1–18) are forms of a particular law which is usually referred to as the *equation of state*. The outlet state of the fluid may, in principle, be determined from Eqs. (1–14), (1–15), (1–16), and (1–17). However, as before, the details are not important and will not be considered here.

It is now clear that in its first formulation, Example 1–4 is a thermodynamically determined problem and can be solved by using the general laws of thermodynamics, combined with those of mechanics. In its second formulation, however, the problem requires the use of an additional particular law, and therefore it is thermodynamically undetermined.

Gas dynamics and *heat transfer* are the major disciplines which deal with thermodynamically undetermined problems. In addition to the general laws of thermodynamics, gas dynamics depends on the equation of state as a particu-

lar law. Heat transfer employs two particular laws, related to the so-called *modes of heat transfer* which we shall now describe.

Diffusion. In diffusion, heat is transferred through a medium or from one to another of two media in contact, if there exists a nonuniform temperature distribution in the medium or between the two media. On the molecular level, the mechanism of diffusion is visualized as the exchange of kinetic energy between the molecules in the regions of high and low temperatures. Particularly, it is attributed to the elastic impacts of molecules in gases, to the motion of free electrons in metals, and to the longitudinal oscillations of atoms in solid insulators of electricity.

Radiation. The true nature of radiation and its transport mechanism have not been completely understood to date. Some of the effects of radiation can be described in terms of electromagnetic waves, and others in terms of quantum mechanics, although neither theory explains all the experimental observations. According to wave theory, for example, during the emission of radiation a body continuously converts part of its internal energy to electromagnetic waves, another form of energy. These waves travel through space with the velocity of light until they strike another body, where part of their energy is absorbed and reconverted into internal energy.

In the foregoing classification we have not considered *convection* to be a mode of heat transfer. Actually, convection is motion of the medium which facilitates heat transfer by diffusion and/or radiation. For customary reasons only, we distinguish between the diffusion of heat in moving or stationary rigid bodies, which we shall call *conduction*, and the diffusion of heat in moving deformable bodies, which we shall call *convection*. Conduction is the subject of this volume; convection and radiation will be treated elsewhere. Nevertheless, examples from convection and radiation are occasionally included in this text when pertinent.

1–2. Continuum Theory Versus Molecular Theory*

In the preceding section the process of heat transfer by diffusion was described in two different ways. From a macroscopic or phenomenological standpoint, heat, as evidenced from experimental observations, is transferred from a region of higher temperature to a region of lower temperature in a medium. From a microscopic or molecular standpoint, the transfer of heat is thought to come about through the exchange of kinetic energy between molecules. This theory, however, is based on hypothesis rather than experiment. The contrast between these two views of heat transfer is reflected in the two alternative approaches to engineering problems in heat transfer.

---

* The continuum theory is also called the *field* or *macroscopic* theory, or *phenomenological* view; the molecular theory is also called the *microscopic* theory.

In the first approach, corresponding to the macroscopic view, the medium is assumed to be a continuum. That is, the mean free path of molecules is small compared with all other dimensions existing in the medium, such that a statistical average (global description) is possible. In other words, the medium fits the definition of the concept of *field*. The properties of a field may be scalar, such as temperature T, or vectorial, such as velocity $\mathbf{V}$. In the second approach, corresponding to the molecular view, either a statistical average of the molecular behavior is not possible or it is possible but not desired. Actually, a very general and logical description of a medium that consists of a spatially distributed molecular structure would be one in which the general laws were written for each separate molecule. Solving the many-particle system in time and space and then relating a required macroscopic concept to molecular behavior would obviously produce the same result as that obtained from the continuum theory.

The reason for not always starting with the molecular approach, apart from the mathematical difficulties and the fact that we actually know little about intermolecular forces, is that the behavior of molecules or small particles of a medium may not be of particular interest. On the contrary, as in most cases of engineering, the problem may be to determine how the medium behaves as a whole—to find, for example, the velocity and/or temperature variations in a medium. Here the convenience of using the field concept becomes clear. Obviously, there are cases in which it is advantageous to use one of the foregoing methods in preference to the other. For example, one would hardly think of solving a conduction problem of a solid body from the particle standpoint, or explaining the behavior of rarefied gases by means of continuum considerations. A large number of problems exists, however, for which both approaches can be used conveniently, the choice depending on previous experience, skill, or taste. From the viewpoint of physical interpretation the only difference is that the averaging process of the molecular structure is undertaken before or after the analysis, depending on the approach used. That is, the statistics either precedes or follows the mechanics (or thermodynamics).

In this text our interest lies not in the individual behavior of the molecules, but rather in their mean effects in space and time. In other words, the problem is to determine how a medium behaves as a whole, or how such parts of it containing a large number of molecules behave. We therefore will be looking at problems of heat transfer from the continuum standpoint.

1–3. Foundations of Continuum Heat Transfer

Any engineering science is based on both theory and experiment. The answer to the question "Why not only experiment or theory rather than both together?" is that each is a tool fundamentally different from the other, and each has its own idealizations and approximations which may not pertain to the other. In the search for reality, both are needed, for reality may be closely approached by cross-checking the results of theoretical and experimental in-

vestigations. Therefore, although the present text is directed toward theoretical heat transfer only, let it be clearly understood that this is a matter of the author's area of competence rather than an indication of the importance of theory as opposed to experiment.

Problem-solving in theoretical heat transfer, as in other disciplines of engineering, may be outlined as follows:

A problem posed by reality
↓
Formulation by idealization
(Physics)
↓
Solution by approximation
(Mathematics)
↓
Interpretation of physical
meaning of answer

From this outline we see that two principles are involved in all problems of engineering, the principle of *idealization* and the principle of *approximation.*

In the formulation of a problem, idealizations are necessary even for the definition of concepts and the statement of natural (general and particular) laws. A well-known example of the idealized definition of concepts is the density b* of a physical property B of a medium at a point P. This is defined according to the following idealized limiting process. A small sphere of radius R is considered at P, then the total quantity of the property ΔB contained in the sphere is divided by its volume ΔV, and R (or ΔV) is allowed to approach zero. The foregoing limiting process is actually an unrealistic idealization. As a matter of fact, when the volume of the sphere becomes less than a value $\Delta V_0(R_0)$ the density changes discontinuously, as a result of the molecular structure of the medium. However, extrapolating the density curve from $\Delta V_0(R_0)$ to $\Delta V(0) = 0$ as shown in Fig. 1–6, we substitute for the discontinuous behavior the continuous behavior and, passing from the difference form to the differential form, we have

$$b = \lim_{\Delta V \to 0} \frac{\Delta B}{\Delta V} = \frac{dB}{dV}, \quad (1\text{–}19)$$

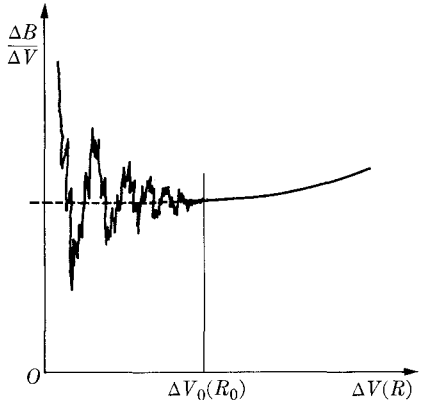


FIG. 1–6

* Here the density is, in the generalized sense, a quantity per unit volume. It may or may not be the mass density.

which is a convenient mathematical definition of density at a point. Well-known examples of applications of this definition are mass density, mass concentration, and electric charge density. The same procedure may be applied to other volume- or mass-dependent properties* of a medium.

The second idealization in the formulation of a problem involves the individual terms appearing in the statement of general laws. This may be illustrated by considering, for example, Newton's second law of motion. The forces described by this law may be categorized as body and surface forces. The body forces acting on a differential volume and the surface forces acting on a differential surface element are both idealized to be identical to a vector by excluding the couples.† Further idealizations may involve the complete omission of certain terms. Without these and many other idealizations, the continuum theory of engineering as used in the mechanics of rigid and deformable bodies, thermodynamics, gas dynamics, heat transfer, or electromagnetics would be impossible to formulate. These fields deal with ideal continua, although the continuous media involved consist of a finite but very large number of discrete individual particles.

Once the natural laws of continuum theory have been established (on the basis of a number of idealizations) we must find ways of formulating the problems posed by reality. The formulation selected depends on our ability to fit problems to the natural laws, a process which often requires further approximations of these laws to make them applicable to the specific problem under consideration.

It should always be kept in mind that a problem may be formulated in a number of approximate ways, and that intuition, insight, and experience are required in order to select the formulation best suited to the problem. Intuition and insight, unfortunately, cannot be taught and depend on the individual. However, the equally important experience can be gained with faithful practice. Similarly, the solution of a formulated problem may be obtained in a number of approximate ways, and again, the selection of the most suitable method of solution requires the intuition, insight, and experience of the individual, although in mathematics rather than in physics. The solution of a well-formulated problem should satisfy the criteria of existence, uniqueness, and stability. Existence and uniqueness, although important, are the concerns of the pure scientist. Stability, on the other hand, is obviously of great importance to the applied scientist.

This text is divided into three parts. Part I deals with the formulation, and Part II with the methods of exact and approximate solutions of conduction problems. Part III is devoted to further formulation and solution methods of an advanced or special nature.

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* These are sometimes called *extensive* properties.

† These and further assumptions are necessary for the formulation of the Euler and Navier-Stokes equations of fluids.

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2. M. H. SHAMOS and G. M. MURPHY, *Recent Advances in Science.* New York: Interscience Publishers, 1956.

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PART I | FORMULATION

LUMPED, INTEGRAL, AND DIFFERENTIAL FORMULATIONS

In Chapter 1 the place of heat transfer among the engineering disciplines was established and the modes of heat transfer—conduction, convection, and radiation—were distinguished. Having this background we now proceed to the general formulation of conduction problems.

The formulation or physics of the analytical phase of an engineering science such as heat transfer is based on *definitions of concepts* and on *statements of natural laws* in terms of these concepts. The natural laws of conduction, like those of other disciplines, can be neither proved nor disproved but are arrived at inductively, on the basis of evidence collected from a wide variety of experiments. As man continues to increase his understanding of the universe, the present statements of natural laws will be refined and generalized. For the time being, however, we shall refer to these statements as the available approximate descriptions of nature, and employ them for the solution of current problems of engineering.

As we saw in Chapter 1, the natural laws may be classified as (1) general laws, and (2) particular laws. *A general law is characterized by the fact that its application is independent of the nature of the medium under consideration.* Examples are the law of conservation of mass, Newton's second law of motion (including momentum and moment of momentum), the first and second laws of thermodynamics, the law of conservation of electric charge, Lorentz's force law, Ampere's circuit law, and Faraday's induction law. The problems of nature which can be formulated completely by using only general laws are called *mechanically, thermodynamically, or electromagnetically determined problems.* On the other hand, the problems which cannot be formulated completely by means of general laws alone are called *mechanically, thermodynamically or electromagnetically undetermined problems.* Each problem of the latter category requires, in addition to the general laws, one or more conditions stated in the form of particular laws. *A particular law is characterized by the fact that its application depends on the nature of the medium under consideration.* Examples are Hooke's law of elasticity, Newton's law of viscosity, the ideal gas law, Fourier's law of conduction, Stefan-Boltzmann's law of radiation, and Ohm's law of electricity.

In this text we shall employ three general laws,

(a) the law of conservation of mass,

(b) the first law of thermodynamics,

(c) the second law of thermodynamics,

and two particular laws,

(d) Fourier's law of conduction,

(e) Stefan-Boltzmann's law of radiation,

each with a different degree of importance. For this reason the first seven sections of this chapter are devoted to a review of these laws and their associated concepts.

2–1. Definition of Concepts

Starting with the hypothesis that the universe is a medium of molecular structure containing energy, let us first define the following concepts.

Continuum (field). A medium in which the smallest volume under consideration contains enough molecules to permit the statistically averaged characteristics to adequately describe the medium. [See, for example, the definition of density b (Eq. 1–19).]

System. A part of a continuum which is separated from the rest of the continuum for convenience in the formulation of a problem. The boundaries of a system may expand or contract, but they are always so assumed that the rest of the continuum does not cross them during any change of the system. The *Lagrangian* method of fluid mechanics is used in the mathematical description of a system.

Control volume. The same as a system, except that the rest of the continuum may cross the fixed or deformable boundaries (control surfaces) of a control volume at one or more places. This is the *only* difference between a control volume and a system. For most problems in this text, control volumes with deformable boundaries are not necessary and, except for simple cases,* are not considered. The *Eulerian* method of fluid mechanics is used in the mathematical description of a control volume.

Property. A macroscopic characteristic of a system or control volume which is ascertained by a statistical averaging procedure. Properties, such as density, velocity, pressure, temperature, internal energy, kinetic energy, potential energy, enthalpy, or entropy, are observed or evaluated quantitatively. The mathematical description of a property B is that between any initial and final conditions the change in B does not depend on the path followed, that is,

$$\int_1^2 dB = B_2 - B_1.$$

State. A condition of a system or control volume which is identified by means of properties. A state may be determined when a sufficient number of independent properties is specified.

Process. A change of any state of a system or control volume.

Cycle. A process whose initial and final states are identical.

Work. A form of energy which is identified as follows. Work is done by a system or control volume on its environment during a process if the system or

* See Section 2–3.

the control volume could pass through the same process while the sole effect external to the system or the control volume was the raising of a weight. *Work done by a system or control volume is considered positive; work done on a system or control volume, negative.* The most common forms of work are discussed in the statements of the first law of thermodynamics. [See the developments of Eqs. (2–16) and (2–36).]

Equality of temperature. When any two systems or control volumes are placed in contact with each other, in general they affect each other as evidenced by changes in their properties. The limiting state which the two approach is called the state of equality of temperature. The definition of equality of temperature implies the existence of states of inequality of temperature for this pair.

Heat. A form of energy which is transferred across the boundaries of a system or control volume during a process by virtue of inequality of temperature. *Heat transferred to a system or control volume is considered positive; heat transferred from a system or control volume, negative.*

It can be shown that the work and heat interactions between a system or control volume and the surroundings depend on the path followed by the associated process. Therefore, *work and heat are not properties.*

The natural laws may now be stated in terms of the foregoing concepts. First let us consider the general laws.

2–2. Statement of General Laws

The first step in the statement of a general law is the selection of a system or control volume. Without this step it is meaningless to speak of such concepts as density, velocity, pressure, temperature, heat, work, internal energy, or entropy, which are the terms used in the statements of general laws. Although the well-known, simple forms of the general laws are always written for a system, system analysis becomes inconvenient when dealing with continua in motion, because it is often difficult to identify the boundaries of a moving system for any appreciable length of time. The control-volume approach is therefore generally preferred for continua in motion.

The second step in the statement of a general law is the selection of the form of this law. A general law may be formulated in either of the following forms:

(1) lumped (or averaged);

(2) distributed: (a) integral, (b) differential, (c) variational, (d) difference.

A general law is said to be lumped if its terms are independent of space, and to be distributed if its terms depend on space. This is demonstrated in Fig. 2–1 by a volume property B. At a point $P(\mathbf{r})$, where $\mathbf{r}$ denotes the position vector of P, $B = B(t)$ and $B = B(P, t)$ denote the lumped and distributed values, respectively, of B.

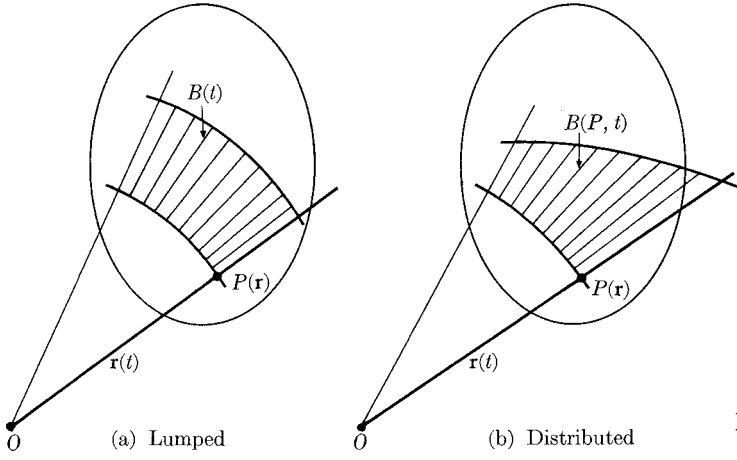


FIG. 2–1

Because of the assumed background of the reader, the development of the variational and difference formulations of the general laws is delayed until Chapters 8, 9, and 10. Here we shall consider only the lumped, integral, and differential formulations of these laws. We shall do this in terms both of a system and of a control volume. It should be noted, however, that there exists a *transformation formula* (Reynold's transport theorem) to convert a general law stated for a system to that for a control volume, thereby eliminating the necessity for separate developments of a general law for both a system and a control volume. It is with mass- or volume-dependent properties that we derive the different forms of this transformation formula suitable to the lumped, integral, and differential formulations of general laws.

2–3. Lumped Formulation of General Laws

First let us develop the form of the transformation formula applicable to the lumped case.

Let B be a mass- or volume-dependent scalar property whose specific value b is

$$b = B/m \qquad \text{or} \qquad B/V. \tag{2–1}$$

The most frequent examples of B and b are listed in Table 2–1. Here we shall derive the transformation formula in terms of the specific value of the property in relation to the mass, $b = B/m$. The same procedure may be readily extended to the volume-dependent case.

Consider a control volume undergoing a process whose initial and final states are given in Fig. 2–2. During this process the mass Δm_i having the lumped specific value b_i of property B crosses the boundaries of the control volume at the location i. (Here Δ denotes a finite change in any property.) The expansion or contraction of the boundaries of the control volume does not represent any

TABLE 2–1

Property	B	$b = B/m$	$b = B/\mathcal{V}$
Mass	m	1	ρ
Volume	$\mathcal{V} = mv$	v	1
	$\mathcal{V} = m/\rho$	$1/\rho$	1
Momentum	mV	V	ρV
Kinetic energy	$\frac{1}{2}mV^2$	$\frac{1}{2}V^2$	$\frac{1}{2}\rho V^2$
Potential energy	mgz	gz	ρgz
Internal energy	$U = mu$	u	ρu
Total energy	$E = me$	e	ρe
Enthalpy	$H = mh$	h	ρh
Entropy	$S = ms$	s	ρs
Mass concentration	$C = mc$	c	ρc
Electric charge	$q_e = \mathcal{V}\rho_e$	ρ_e/ρ	ρ_e

difficulty and is therefore included in the discussion.* Consider at the same time a system that coincides with the control volume at the final state, but at the initial state occupies the volume $\Delta\mathcal{V}_i$ as well as the control volume. We wish to find the relation between the changes in the property B within the system and within the control volume.

Let B_1, B_2, and B', B'' denote the initial and final values of B for the system and the control volume, respectively. During the process shown in Fig. 2–2

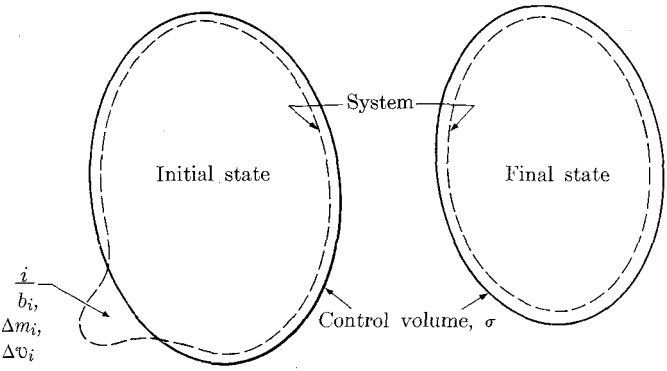


FIG. 2–2

* The initial and final states of Fig. 2–2 are shown separately for clarity. While the boundaries are deforming, the whole control volume may or may not be stationary.

the change in B of the system is

$$\Delta B = B_2 - B_1. \tag{2–2}$$

Referring again to Fig. 2–2 and expressing B_1 and B_2 in terms of the values for the control volume, we have

$$B_1 = B' + b_i \Delta m_i, \qquad B_2 = B''. \tag{2–3}$$

Then, introducing Eq. (2–3) into Eq. (2–2) and denoting the change within the control volume by ΔB_σ, we have

$$\Delta B = \Delta B_\sigma - b_i \Delta m_i. \tag{2–4}$$

If the property B crosses the control volume at more than one place, Eq. (2–4) becomes

$$\Delta B = \Delta B_\sigma - \sum_{i=1}^{N} b_i \Delta m_i, \tag{2–5}$$

which is the desired transformation formula. Here N is the number of crossings, and positive Δm_i indicates flow to the control volume. Finally, dividing each term of Eq. (2–5) by Δt* and carrying out the limiting process $\Delta t \to 0$, we find the transformation formula to be

$$\underbrace{\frac{dB}{dt}}_{\text{system}} = \underbrace{\frac{dB_\sigma}{dt} - \sum_{i=1}^{N} b_i w_i}_{\text{control volume}}, \tag{2–6}$$

where $w_i = \lim_{\Delta t \to 0} (\Delta m_i / \Delta t)$ is the mass flow rate crossing the control volume at the location i.

Let us now use the transformation formula, Eq. (2–6), to obtain the lumped forms of the general laws for control volumes.

Conservation of mass (lumped formulation). By definition, a system is so constituted that no continuum (mass) may cross its boundaries; therefore, for a system we have

$$\frac{dm}{dt} = 0, \tag{2–7}$$

which is the equation for *conservation of mass for lumped systems.* The conservation of mass for the control volume of Fig. 2–2 may now be obtained from Eqs. (2–6) and (2–7). Noting from Table 2–1 that

$$B = m, \qquad b = 1, \tag{2–8}$$

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* Although time is neither a property nor a nonproperty of a system or control volume, hereafter any change in time will be denoted by the same symbol as that used for a change in a property.

and introducing Eq. (2-8) into Eq. (2–6), we have

$$\frac{dm}{dt} = \frac{dm_\sigma}{dt} - \sum_{i=1}^{N} w_i.$$

From this result and Eq. (2–7) it follows that

$$0 = \frac{dm_\sigma}{dt} - \sum_{i=1}^{N} w_i. \tag{2–9}$$

This is the equation for *conservation of mass for lumped control volumes.*

First law of thermodynamics (lumped formulation). Since this law is important from the standpoint of heat transfer it will be developed in detail.

Across the boundaries of a system which completes a cycle the net heat is proportional to the net work:

$$\nabla Q = \nabla W, \tag{2–10}$$

where ∇ denotes the net amount of a nonproperty, Q the heat, and W the work.

If the system undergoes a process, instead, the difference between the net heat and work is equal to the change of the *total energy* (a property) of the system:

$$\Delta E = \nabla Q - \nabla W. \tag{2–11}$$

Here E might be present in a variety of forms, such as internal, kinetic, potential, chemical, and nuclear energy. Thus

$$E = U + \tfrac{1}{2}mV^2 + mgz + U_{\text{chem}} + U_{\text{nucl}}. \tag{2–12}$$

The rate form of Eq. (2–11) gives the *first law of thermodynamics for lumped systems:*

$$\frac{dE}{dt} = q - P, \tag{2–13}$$

where $q = \lim_{\Delta t \to 0} (\nabla Q/\Delta t)$ denotes the rate of heat transfer and $P = \lim_{\Delta t \to 0} (\nabla W/\Delta t)$ the rate of work (power). Note that Eq. (2–13), in terms of our classification of general laws, is the spatially lumped but timewise differential form of the first law of thermodynamics for systems.

Applying the transformation formula of Eq. (2–6) in the usual manner, that is, noting from Table 2–1 that

$$B = E, \qquad b = e, \tag{2–14}$$

then introducing Eq. (2–14) into Eq. (2–6) and the result into Eq. (2–13), we find the rate form of the *first law of thermodynamics for lumped control volumes:*

$$\frac{dE_\sigma}{dt} - \sum_{i=1}^{N} e_i w_i = q - P, \tag{2–15}$$

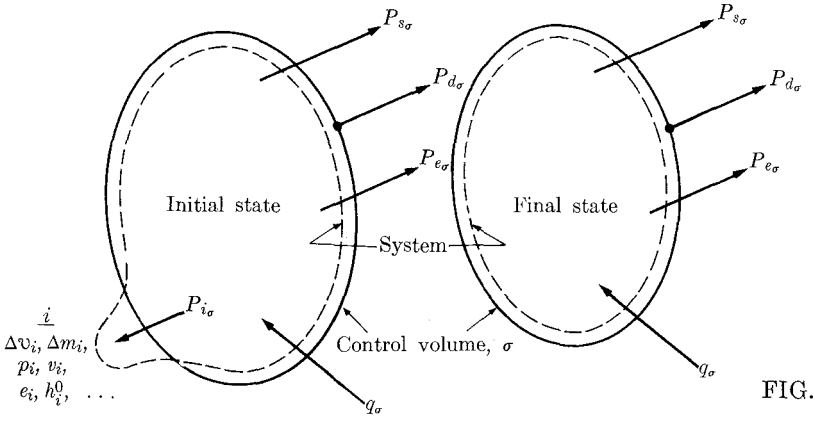


FIG. 2–3

where P is the power leaving and q the rate at which heat is received by the system.

The power P is usually composed of three terms,

$$P = P_d + P_s + P_e,$$

P_d, P_s, and P_e being the displacement, shaft, and electrical powers, respectively. These powers may be conveniently expressed in terms of those leaving the control volume as follows:

$$P_d = (P_d + P_i)_\sigma, \qquad P_s = P_{s_\sigma}, \qquad P_e = P_{e_\sigma} + P_{e_{\Delta v,i}},$$

where P_{d_σ}, P_{i_σ}, P_{s_σ}, and P_{e_σ} denote the displacement, mass flow, shaft, and electrical powers, respectively (Fig. 2–3).

Assume, for example, that the total internal energy is generated at the rate of

$$U' = U'_\sigma + U'_{\Delta v_i}$$

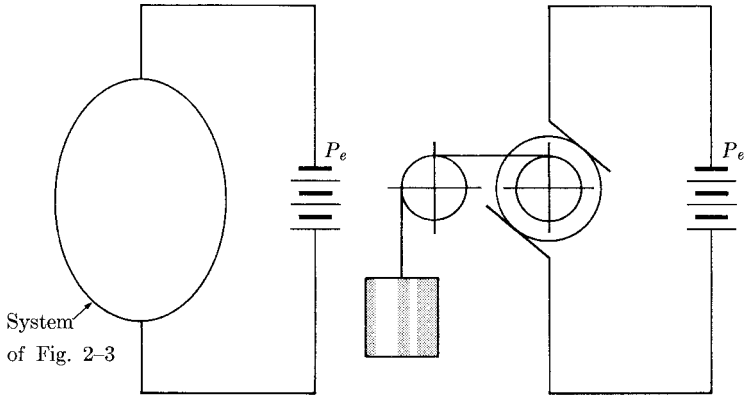
in the system of Fig. 2–3 by means of an external electric circuit (Fig. 2–4a). According to the definition of work, this energy generation is equivalent to the power P_e drawn to the system (Fig. 2–4b). Thus we have

$$P_e = U',$$

where

$$P_e = P_{e_\sigma} + P_{e_{\Delta v,i}}.$$

In the literature the expression "heat generation" is erroneously employed in place of internal energy generation.



(a) Actual circuit (b) Equivalent circuit FIG. 2–4

The electrical power $P_{e_{\Delta\mathcal{V},i}}$, being identical to the electrical internal energy generated in the volume $\Delta\mathcal{V}_i$, must be included in the internal energy flow

$$\sum_{i=1}^{N} e_i w_i$$

across the boundaries of the control volume. If the *friction* is *negligible*, the mass flow power P_{i_σ} reduces to

$$P_{i_\sigma} = -\lim_{\Delta t \to 0} \sum_{i=1}^{N} (p_i \, \Delta\mathcal{V}_i/\Delta t) = -\sum_{i=1}^{N} p_i v_i w_i,$$

where $\Delta\mathcal{V}_i$ is the volume and v_i the specific volume of the mass Δm_i, and p_i is the pressure on the boundaries of $\Delta\mathcal{V}_i$.

Similarly, the rate of heat q received by the system may be conveniently expressed in terms of q_σ received by the control volume. Thus we have

$$q = q_\sigma + q_{\Delta\mathcal{V}_i}.$$

However, from the standpoint of control-volume analysis $q_{\Delta\mathcal{V}_i}$, like P_{i_σ}, may be considered as an internal energy increase in the volume $\Delta\mathcal{V}_i$ and included in the internal energy flow $\sum_{i=1}^{N} e_i w_i$.

Rearranging Eq. (2–15) according to the foregoing discussion on the power and heat terms, we obtain the explicit form of the *first law of thermodynamics for lumped control volumes:*

$$\frac{dE_\sigma}{dt} = \sum_{i=1}^{N} h_i^0 w_i + q_\sigma - (P_d + P_s + P_e)_\sigma, \tag{2–16}$$

where $h_i^0 = e_i + p_i v_i$ defines the *stagnation enthalpy* per unit mass.

Second law of thermodynamics (lumped formulation). Let us first write the *second law of thermodynamics for lumped systems,*

$$\Delta S \geq \nabla Q/T, \tag{2–17}$$

in the rate form

$$dS/dt \geq q/T. \tag{2–18}$$

Then, from Table 2–1, inserting

$$B = S, \qquad b = s \tag{2–19}$$

into the transformation formula, Eq. (2–6), and the result into Eq. (2–18), we obtain the *second law of thermodynamics for lumped control volumes:*

$$\frac{dS_\sigma}{dt} \geq \sum_{i=1}^{N} s_i w_i + \frac{q}{T}. \tag{2–20}$$

Introducing the rate of *entropy generation** S_g, we may write Eq. (2–20) as an equality in the form

$$\frac{dS_\sigma}{dt} = \sum_{i=1}^{N} s_i w_i + \frac{q}{T} + S_g, \tag{2–21}$$

which gives the *conservation of entropy for lumped control volumes.* Since S_g is related to the *degree of irreversibility,*† Eq. (2–21) becomes useful for the study of irreversible processes. This point will be elaborated in the differential formulation. [See Eqs. (2–66) through (2–69).]

Having developed the lumped general laws for systems and control volumes, we now proceed to the corresponding integral forms. The general laws stated for control volumes become identical to those for systems as the mass flow across the boundaries approaches zero; therefore, although the starting place will always be the system, we shall develop the integral and differential forms for control volumes only.

2–4. Integral Formulation of General Laws

As with the lumped case, our first step will be the derivation of the appropriate transformation formula. Because of the increased complexity of the integral formulation, however, we shall develop the formula for a *control volume fixed in space.* This proves to be quite satisfactory for most applications of the integral formulations of the general laws.

Consider a control volume which is fixed in space and through which a property B flows (Fig. 2–5). Assume that at time t a system coincides with this control volume. We wish to evaluate the time rate of change of the property within the system.

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* See the concept of *lost work* in Reference 9, Eq. (7–8).

† As $S_g \to 0$ Eq. (2–21) applies to reversible processes only.

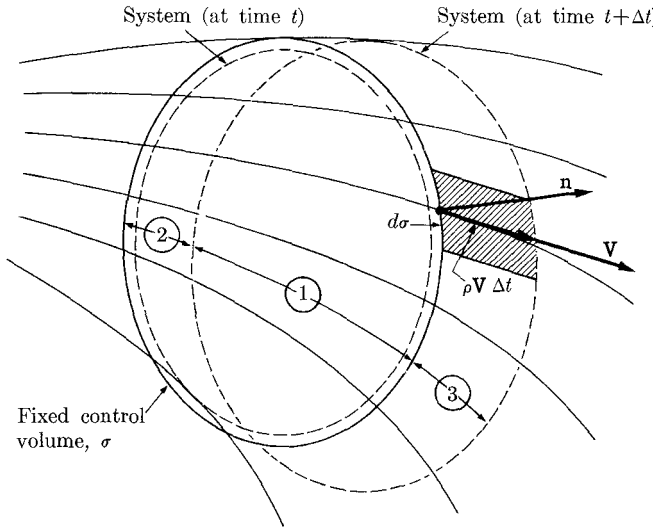


FIG. 2–5

Since the continuum cannot cross its boundaries, this system during a time interval Δt moves with the continuum and at time $t + \Delta t$ occupies another volume in space as shown in Fig. 2–5. Thus the required rate of change may be written in the form

$$\left(\frac{dB}{dt}\right)_{\text{system}} = \lim_{\Delta t \to 0} \frac{[B_1(t + \Delta t) + B_3(t + \Delta t)] - [B_1(t) + B_2(t)]}{\Delta t}, \tag{2–22}$$

or, by rearranging its terms,

$$\left(\frac{dB}{dt}\right)_{\text{system}} = \lim_{\Delta t \to 0} \left[\underbrace{\frac{B_1(t + \Delta t) - B_1(t)}{\Delta t}}_{\text{first term}} + \underbrace{\frac{B_3(t + \Delta t)}{\Delta t}}_{\text{second term}} - \underbrace{\frac{B_2(t)}{\Delta t}}_{\text{third term}}\right], \tag{2–23}$$

where B_1, B_2, and B_3 are the instantaneous values of B corresponding to three regions of space, denoted by 1, 2, and 3 in Fig. 2–5.

As $\Delta t \to 0$ the space 1 coincides with the control volume, and the first term on the right of Eq. (2–23) gives the time rate of change of B within the control volume. Because of the fact that B now depends on space as well as time, this rate of change may be conveniently expressed in terms of $B = \int_{\mathcal{V}} b\rho \, d\mathcal{V}$, $\mathcal{V}$ being the fixed volume of the control volume, $d\mathcal{V}$ the volume element, and $b = B/m$. Thus we have for the first term of Eq. (2–23)*

$$\lim_{\Delta t \to 0} \frac{B_1(t + \Delta t) - B_1(t)}{\Delta t} = \left(\frac{dB}{dt}\right)_{\mathcal{V}} = \int_{\mathcal{V}} \frac{\partial(b\rho)}{\partial t} \, d\mathcal{V}. \tag{2–24}$$

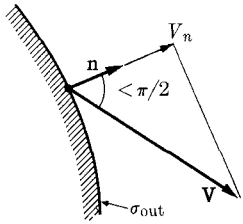
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* Since the control volume is fixed, $d/dt \equiv \partial/\partial t$, and the order of differentiation and integration is interchangeable.

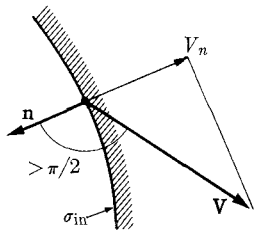
The second and third terms in Eq. (2–23) respectively give the outgoing and incoming flows of B through the control surface σ. These terms may be evaluated by considering the shaded cylinder shown in Fig. 2–5. The height of this cylinder is $(\mathbf{V}\,\Delta t)\cdot\mathbf{n}$, the volume $(\mathbf{V}\,\Delta t)\cdot\mathbf{n}\,d\sigma$, and the mass $\rho(\mathbf{V}\,\Delta t)\cdot\mathbf{n}\,d\sigma$; $\mathbf{V}$ denotes the velocity of the flow, $\mathbf{n}$ the outward normal vector, and $d\sigma$ the surface element of the control volume. Thus the flow rates of the mass and of the property B through $d\sigma$ are found to be $\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma$ and $b\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma$, respectively. Integrating the latter over the entire control surface, we get

$$\lim_{\Delta t\to 0}\left[\frac{B_3(t+\Delta t)}{\Delta t}-\frac{B_2(t)}{\Delta t}\right]=\int_\sigma b\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma. \tag{2–25}$$

The explicit form of Eq. (2–25) for outgoing and incoming flows is illustrated in Fig. 2–6.



$$\lim_{\Delta t\to 0}\frac{B_3(t+\Delta t)}{\Delta t}=\int_\sigma b\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma$$
$$=\int_{\sigma_{\text{out}}} b\rho V_n\,d\sigma_{\text{out}},$$



$$\lim_{\Delta t\to 0}\frac{B_2(t)}{\Delta t}=\int_\sigma b\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma$$
$$=-\int_{\sigma_{\text{in}}} b\rho V_n\,d\sigma_{\text{in}}$$

FIG. 2–6

Finally, introducing Eqs. (2–24) and (2–25) into Eq. (2–23), we find the desired transformation formula to be*

$$\left(\frac{dB}{dt}\right)_{\text{system}}=\underbrace{\int_{\mathcal{V}}\frac{\partial(b\rho)}{\partial t}\,d\mathcal{V}+\int_\sigma b\rho\mathbf{V}\cdot\mathbf{n}\,d\sigma}_{\text{control volume}}. \tag{2–26}$$

It is worth noting that the foregoing transformation formula, as just derived, is based on the physical interpretation of the definitions of system and control volume. The same result might also be obtained from the mathematical de-

* Note that by convention the signs for the flow terms of the lumped and integral forms of the transformation formula are opposite.

scription of these, namely, the Lagrangian and Eulerian representations of the continuum. This mathematical description will not be given here.*

Let us now use Eq. (2–26) to establish the integral form of the general laws for control volumes, starting from the statement of these laws as applied to systems.

Conservation of mass (integral formulation). Introducing the previously used appropriate values

$$B = m, \qquad b = 1 \tag{2–8}$$

into Eq. (2–26) gives

$$\left(\frac{dm}{dt}\right)_{\text{system}} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{\sigma} \rho \mathbf{V} \cdot \mathbf{n}\, d\sigma. \tag{2–27}$$

Since the left-hand side of Eq. (2–27) is equal to zero, we have

$$0 = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{\sigma} \rho \mathbf{V} \cdot \mathbf{n}\, d\sigma, \tag{2–28}$$

the equation for *conservation of mass for integral control volumes.*

First law of thermodynamics (integral formulation). Let us develop the integral form of the first law of thermodynamics for the control volume of Fig. 2–5, starting with the system shown in the figure. The rate form of the first law for this system is

$$(dE/dt)_{\text{system}} = (\delta Q/dt)_{\text{system}} - (\delta W/dt)_{\text{system}}, \tag{2–29}$$

where d and δ denote the differential change in a property and the amount of a nonproperty, respectively.†

Referring to

$$B = E, \qquad b = e \tag{2–14}$$

and the transformation formula given by Eq. (2–26), we may rearrange Eq. (2–29) as follows:

$$\left(\frac{dE}{dt}\right)_{\text{system}} = \int_{\mathcal{V}} \frac{\partial (e\rho)}{\partial t}\, d\mathcal{V} + \int_{\sigma} e\rho \mathbf{V} \cdot \mathbf{n}\, d\sigma. \tag{2–30}$$

Since at time t the system occupies the control volume, the rate of heat transfer $(\delta Q/dt)_{\text{system}}$ and the rate of work $(\delta W/dt)_{\text{system}} = P$ (power) across the boundaries of the system may be conveniently expressed in terms of the control volume.

---

* See, for example, Reference 12, p. 84.

† The order of the amount δ is the same as that of the differential change d.

Thus, introducing the *heat flux** vector $\mathbf{q}$ (Fig. 2–7) and integrating the rate of heat transfer $\mathbf{q} \cdot \mathbf{n}\, d\sigma$ from the surface element $d\sigma$ over the entire control surface, we have

$$(\delta Q/dt)_{\text{system}} = -\int_\sigma \mathbf{q} \cdot \mathbf{n}\, d\sigma, \tag{2–31}$$

where the negative sign is in accordance with the sign convention for heat.

The rate of work $(\delta W/dt)_{\text{system}}$ may be conveniently discussed, as it was in the lumped case, by considering its components. These components are the rates of work done by

(a) the system on the surrounding pressure,

$$(\delta W/dt)_d = P_d,$$

(b) the system on the surrounding viscous stresses,

$$(\delta W/dt)_\tau = P_\tau, \tag{2–32}$$

(c) the shaft of the system on the surroundings,

$$(\delta W/dt)_s = P_s, \tag{2–33}$$

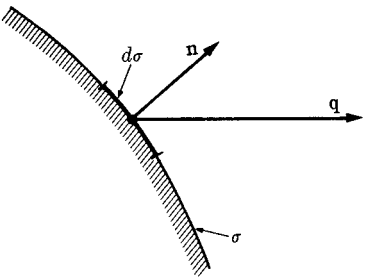


FIG. 2–7

(d) the electrical power of the system on the surroundings,

$$(\delta W/dt)_e = P_e.$$

During the time interval Δt the work done by the system against the pressure on the surface element $d\sigma$ is $p\, d\sigma(\mathbf{V}\, \Delta t) \cdot \mathbf{n}$, where $(\mathbf{V}\, \Delta t) \cdot \mathbf{n}$, the height of the cylinder, is the distance moved normal to $d\sigma$. The rate of this work, $p\mathbf{V} \cdot \mathbf{n}\, d\sigma$, integrated over the entire control surface gives

$$P_p = \int_\sigma p\mathbf{V} \cdot \mathbf{n}\, d\sigma = \int_\sigma (p/\rho)\rho\mathbf{V} \cdot \mathbf{n}\, d\sigma. \tag{2–34}$$

The second integral of Eq. (2–34), obtained by multiplying and dividing the integrand of the first integral by ρ, becomes more convenient when the definition of enthalpy is used. [See Eqs. (2–36) and (2–37).]

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* The definitions of *flow* and *flux* are clearly distinguished in this text. *Flow* is employed to indicate the convection (motion) of a mass- or volume-dependent property across a surface. Examples are mass, momentum, energy, enthalpy, and entropy transfer by convection, and convective electric current. *Flux*, on the other hand, is used for the rate of diffusion per unit area of a property (or nonproperty) through a surface due to the motion of molecules. This includes mass, momentum, and heat transfer by diffusion, and conductive electric current. The notation employed for conduction heat transfer terms is as follows: heat transfer, Q (Btu); rate of heat transfer, q (Btu/hr); and rate of heat transfer per unit area, q with a subscript or superscript, such as q'', q_n, q_x, . . . , or $\mathbf{q}$ (Btu/ft^2 · hr).

For the case in which P_e denotes the power drawn to the system from an external electric circuit, we have

$$P_e = -\int_{\mathcal{V}} u''' \, d\mathcal{V}, \tag{2–35}$$

where u''' is the *rate of local internal energy per unit volume*, generated electrically in the system.

Introducing Eqs. (2–30), (2–31), (2–32), (2–33), (2–34), and (2–35) into Eq. (2–29) gives

$$\int_{\mathcal{V}} \frac{\partial(e\rho)}{\partial t} \, d\mathcal{V} + \int_{\sigma} e\rho \mathbf{V} \cdot \mathbf{n} \, d\sigma = -\int_{\sigma} \mathbf{q} \cdot \mathbf{n} \, d\sigma - \int_{\sigma} \frac{p}{\rho} \rho \mathbf{V} \cdot \mathbf{n} \, d\sigma - P_\tau - P_s + \int_{\mathcal{V}} u''' \, d\mathcal{V}. \tag{2–36}$$

Furthermore, by employing the definition of *stagnation enthalpy* per unit mass, $h^0 = e + pv$, Eq. (2–36) may be rearranged to yield the *first law of thermodynamics for integral control volumes:*

$$\int_{\mathcal{V}} \frac{\partial(e\rho)}{\partial t} \, d\mathcal{V} + \int_{\sigma} h^0 \rho \mathbf{V} \cdot \mathbf{n} \, d\sigma = -\int_{\sigma} \mathbf{q} \cdot \mathbf{n} \, d\sigma - P_\tau - P_s + \int_{\mathcal{V}} u''' \, d\mathcal{V}. \tag{2–37}$$

Second law of thermodynamics (integral formulation). The rate form of this law written for the system of Fig. 2–5 is

$$dS/dt \geq q/T. \tag{2–18}$$

The right-hand side of Eq. (2–18) may be expressed in terms of the heat flux vector $\mathbf{q}$ as

$$q/T = -\int_{\sigma} (\mathbf{q}/T) \cdot \mathbf{n} \, d\sigma. \tag{2–38}$$

Inserting

$$B = S, \qquad b = s \tag{2–19}$$

into Eq. (2–26), and then the result, together with Eq. (2–38), into Eq. (2–18) yields the *second law of thermodynamics for integral control volumes:*

$$\int_{\mathcal{V}} \frac{\partial(s\rho)}{\partial t} \, d\mathcal{V} + \int_{\sigma} s\rho \mathbf{V} \cdot \mathbf{n} \, d\sigma \geq -\int_{\sigma} \left(\frac{\mathbf{q}}{T}\right) \cdot \mathbf{n} \, d\sigma. \tag{2–39}$$

As with Eq. (2–21) of the lumped case, by adding the total entropy generation

$$S_g = \int_{\mathcal{V}} s''' \, d\mathcal{V}$$

to the right-hand side of Eq. (2–39) we obtain the equation for *conservation of entropy for integral control volumes:*

$$\int_{\mathcal{V}} \frac{\partial(s\rho)}{\partial t}\, d\mathcal{V} + \int_{\sigma} s\rho \mathbf{V} \cdot \mathbf{n}\, d\sigma = -\int_{\sigma} \left(\frac{\mathbf{q}}{T}\right) \cdot \mathbf{n}\, d\sigma + \int_{\mathcal{V}} s'''\, d\mathcal{V}, \quad (2\text{–}40)$$

where s''' is the *rate of local entropy generation per unit volume.*

Having finished the study of the integral forms of the general laws, we now proceed to the differential forms of these laws.

2–5. Differential Formulation of General Laws

There are two ways of obtaining the differential forms of the general laws. One of these starts with the integral form and, using the divergence (Green's) theorem, converts the surface integrals of this form to the volume integrals; then omitting the volume-integral operation results in the differential form. The second method directly establishes the differential forms in terms of an appropriately chosen differential control volume. The former method, being shorter for the present discussion, is preferred here.

We first note the divergence theorem, which states that for a volume $\mathcal{V}$ enclosed by a piecewise smooth surface σ,

$$\int_{\sigma} \mathbf{a} \cdot \mathbf{n}\, d\sigma = \int_{\mathcal{V}} \nabla \cdot \mathbf{a}\, d\mathcal{V}, \quad (2\text{–}41)$$

where $\mathbf{a}$ is any continuously differentiable vector. Then by employing Eq. (2–41), the differential forms of the general laws may be obtained from their integral forms.

Conservation of mass (differential formulation). The integral form of this law, Eq. (2–28), after its surface integral is converted into a volume integral by using Eq. (2–41) with $\mathbf{a} = \rho\mathbf{V}$, may be rearranged in the form

$$0 = \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})\right] d\mathcal{V}. \quad (2\text{–}42)$$

Since Eq. (2–42) is true for an arbitrary control volume, the integrand itself must vanish everywhere, thus yielding the *conservation of mass for differential control volumes:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \quad (2\text{–}43)$$

Noting the well-known vectorial identity

$$\nabla \cdot (\alpha \boldsymbol{\beta}) = \alpha \nabla \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \nabla \alpha \quad (2\text{–}44)$$

(where α is a scalar, $\boldsymbol{\beta}$ is a vector) and the definition of *derivative following the motion,**

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla} \alpha, \tag{2–45}$$

we find that an alternative form of Eq. (2–43) is

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{V} = 0. \tag{2–46}$$

For solids and incompressible fluids $\rho = \text{const}$, and Eq. (2–46) reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0. \tag{2–47}$$

Before considering the other general laws, let us rearrange the integral form of the transformation formula, Eq. (2–26), in the light of the differential form of the law of conservation of mass. Converting the surface integral of Eq. (2–26) into a volume integral, and employing Eqs. (2–44) and (2–43), respectively, we have

$$\left(\frac{dB}{dt}\right)_{\text{system}} = \int_{\mathcal{V}} \rho \frac{db}{dt} \, d\mathcal{V}. \tag{2–48}$$

Hereafter this form of the transformation formula will be used for the differential formulation of general laws.

First law of thermodynamics (differential formulation). The left-hand side, Eq. (2–30), of the integral form of the first law of thermodynamics, Eq. (2–36), may readily be written by transforming the left-hand side of Eq. (2–30) by Eq. (2–48). Thus we have

$$\begin{aligned} \int_{\mathcal{V}} \rho \frac{de}{dt} \, d\mathcal{V} = &- \int_{\sigma} \mathbf{q} \cdot \mathbf{n} \, d\sigma - \int_{\sigma} p\mathbf{V} \cdot \mathbf{n} \, d\sigma \\ &- P_\tau - P_s + \int_{\mathcal{V}} u''' \, d\mathcal{V}. \end{aligned} \tag{2–49}$$

Since our interest lies in solids and in frictionless incompressible fluids, $P_\tau = 0$. Furthermore, considering the special case where $P_s = 0$, and converting the surface integrals related to the heat flux and the rate of pressure work to volume integrals, then eliminating the volume-integral operation, we get

$$\rho \frac{de}{dt} + \boldsymbol{\nabla} \cdot (p\mathbf{V} + \mathbf{q}) = u'''. \tag{2–50}$$

---

* This derivative is often denoted by D/Dt, and is commonly referred to as the *material derivative, substantial derivative,* or *convective derivative.*

Expanding $\nabla \cdot (p\mathbf{V})$ by Eq. (2–44), and for $\rho = \text{const}$, using Eq. (2–47), we get

$$\nabla \cdot (p\mathbf{V}) = \mathbf{V} \cdot \nabla p. \tag{2–51}$$

Inserting Eq. (2–51) into Eq. (2–50) gives

$$\rho \frac{de}{dt} + \mathbf{V} \cdot \nabla p + \nabla \cdot \mathbf{q} = u''', \tag{2–52}$$

which is the *first law of thermodynamics for differential control volumes.* Since Eq. (2–52) is a statement of the law of *conservation of total energy*, we may deduce a number of results from it.

In the absence of thermal, chemical, and nuclear effects, Eq. (2–52) reduces to the law of *conservation of mechanical energy for differential control volumes:*

$$\rho \frac{d}{dt}(\tfrac{1}{2}V^2 + gz) + \mathbf{V} \cdot \nabla p = 0. \tag{2–53}$$

Equation (2–53), being directly available from Newton's second law of motion, applies equally to processes involving thermal, chemical, and/or nuclear effects.

For a steady process, the derivative of p following the motion is

$$\frac{dp}{dt} = \mathbf{V} \cdot \nabla p. \tag{2–54}$$

Introducing Eq. (2–54) into Eq. (2–53) and rearranging terms yields

$$\frac{d}{dt}\left(\frac{p}{\rho} + \frac{1}{2}V^2 + gz\right) = 0, \tag{2–55}$$

which integrates to *Bernoulli's equation* along a streamline:

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{const}. \tag{2–56}$$

The difference between Eqs. (2–53) and (2–52) gives the law of *conservation of thermal energy for differential control volumes:*

$$\rho \frac{du}{dt} + \nabla \cdot \mathbf{q} = u'''. \tag{2–57}$$

Equation (2–57) may further be rearranged in the light of thermodynamic considerations. A state of any property of a continuum which is homogeneous and invariable in composition (but which may be a mixture of two phases) is completely determined by two independent properties.* Thus, considering $u(v, T)$ and $h(p, T)$, and introducing the definitions of *specific heat at constant volume*

* This is the pure substance of thermodynamics. See, for example, Reference 9, p. 30.

and *specific heat at constant pressure,*

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v, \qquad c_p = \left(\frac{\partial h}{\partial T}\right)_p,$$

we have

$$du = \left(\frac{\partial u}{\partial v}\right)_T dv + c_v\, dT, \tag{2–58}$$

$$dh = \left(\frac{\partial h}{\partial p}\right)_T dp + c_p\, dT. \tag{2–59}$$

For solids and incompressible fluids v = const, and Eq. (2–58) reduces to

$$du = c_v\, dT. \tag{2–60}$$

Furthermore, if p = const, Eq. (2–59) becomes

$$dh = c_p\, dT, \tag{2–61}$$

and from the definition of *enthalpy* per unit mass, $h = v + pv$, we get

$$dh = du, \qquad p = \text{const}, \quad v = \text{const}. \tag{2–62}$$

Thus combining Eqs. (2–60), (2–61), and (2–62) gives

$$c_p = c_v = c. \tag{2–63}$$

If p varies, $c_p - c_v$ is negligibly small for solids and incompressible fluids,* and Eq. (2–63) still holds, although only approximately.

Introducing Eq. (2–63) into Eq. (2–60) and the result into Eq. (2–57), we have finally

$$\rho c \frac{dT}{dt} + \nabla \cdot \mathbf{q} = u''', \tag{2–64}$$

as an alternative statement of the law of *conservation of thermal energy for differential control volumes.* In Section 2–7, Eq. (2–64) will be the starting point in deriving the differential form of the equation of conduction.

Second law of thermodynamics (differential formulation). Converting the left-hand side of the integral form of this law, Eq. (2–39), by the transformation formula, Eq. (2–48), and the right-hand side by the divergence theorem, Eq. (2–41), and eliminating the volume-integral operation, we get

$$\rho \frac{ds}{dt} + \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) \geq 0, \tag{2–65}$$

which is the *second law of thermodynamics for differential control volumes.*

---

* See, for example, Reference 9, p. 413, Example 14–4.

Similarly, the integral form of the conservation of entropy, Eq. (2–40), readily yields the *conservation of entropy for differential control volumes:*

$$\rho \frac{ds}{dt} + \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) = s'''. \tag{2–66}$$

Equation (2–66) becomes useful in the evaluation of s''' in terms of $\mathbf{q}$ and T alone. First, from the thermodynamics relation

$$du = T\,ds - p\,dv,$$

we have its rate form for $v = \text{const}$

$$\frac{du}{dt} = T\frac{ds}{dt}. \tag{2–67}$$

Now the elimination of du/dt and ds/dt among Eqs. (2–57), (2–66), and (2–67) gives

$$s''' = \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{1}{T}(\nabla \cdot \mathbf{q}) + \frac{u'''}{T}. \tag{2–68}$$

After we expand $\nabla \cdot (\mathbf{q}/T)$ by Eq. (2–44), Eq. (2–68) may be reduced to

$$s''' = -\frac{1}{T^2}(\mathbf{q} \cdot \nabla T) + \frac{u'''}{T}. \tag{2–69}$$

Equation (2–69) gives, for a differential control volume, the *rate of entropy generation per unit volume* in conduction problems. [When the effect of friction is included in the incompressible fluid, the dissipation of friction causes an increase in entropy generation which is reflected by an additional term in Eq. (2–69). This term, being of little interest for the present study, is not given here.]

We have thus completed the lumped, integral, and differential formulations of the general laws. As preparation for the next section, it will be helpful to focus our attention for a moment on the objective of the study of conduction, which is the design of thermal devices, for example, heat exchangers. This objective cannot be accomplished without evaluating the *temperature* of and *heat transfer* to or from such a device. The temperature is important to the *mechanical design,* since it enables us to calculate *thermal deformation and stress.* The rate of heat transfer is important to the *thermal design,* since it helps us to determine the size of the device, say the heat transfer area of heat exchangers. Thus a heat transfer problem, in general, requires the simultaneous evaluation of $\mathbf{q}$ and T. Keeping this in mind, let us reconsider, for example, the conservation of thermal energy, Eq. (2–64). This equation gives a relation between but is not sufficient for the evaluation of $\mathbf{q}$ and T. Therefore, we are forced to find other equations relating $\mathbf{q}$ to T. Experimental observations show that the additional relations needed are dependent on the continuum under consideration; hence the equations expressing such relations are statements of particular laws, which will now be discussed.

2–6. Statement of Particular Laws

Two particular laws, *Fourier's law of conduction* and *Stefan-Boltzmann's law of radiation*, are considered in this section. The former, being directly related to conduction, is emphasized. The latter becomes useful only when radiation governs the heat transfer from the boundaries of the continuum under consideration.

Fourier's law of conduction. Microscopic theories such as the kinetic theory of gases and the free-electron theory of metals have been developed to the point where they can be used to predict conduction through media. However, the macroscopic or continuum theory of conduction, which is the subject matter of this text, disregards the molecular structure of continua. Thus conduction is taken to be phenomenological and its effects are determined by experiment as follows.

Consider a solid flat plate of thickness L (Fig. 2–8). Part of this plate is assumed to be bounded by an imaginary cylinder of small cross section A and whose axis is normal to the surfaces of the plate. This cylinder is supposed to be so far from the ends of the plate that no heat crosses its peripheral surface; that is, the transfer of heat is one-dimensional along the axis of the cylinder. Let the temperatures of the surfaces of the plate be T_1, T_2 and let us assume, for example, that $T_1 > T_2$. According to the first law of thermodynamics, under steady conditions there must be a constant rate of heat q through any cross section of the cylinder parallel to the surfaces of the plate. From the second law of thermodynamics we know that the direction of this heat is from the higher temperature to the lower.

Experimental observations of different solids lead us to the fact that for sufficiently small values of the temperature difference between the surfaces of the plate,

$$q = kA \frac{T_1 - T_2}{L}, \qquad (2\text{–}70)$$

where k is a constant, the so-called *thermal conductivity* of the material of the plate. Thus the heat flux due to conduction is found to be

$$q_n = k \frac{T_1 - T_2}{L}, \qquad (2\text{–}71)$$

where the subscript of q_n indicates the direction of this flux. Equation 2–71 gives *Fourier's law for homogeneous isotropic continua.*

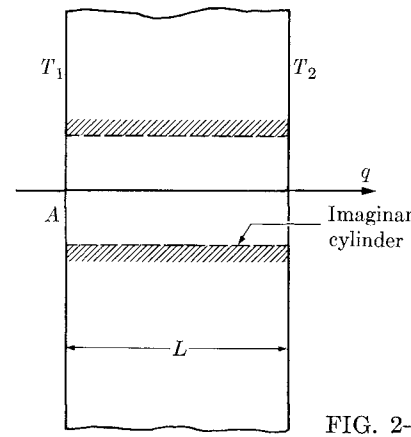


FIG. 2–8

Equation (2–70) may also be used for a fluid (liquid or gas) placed between two plates a distance L apart, provided that suitable precautions are taken to

eliminate convection and radiation. Equation (2–70) therefore describes the conduction of heat in fluids as well as in solids.

Let us now note that continua may be classified according to variations in thermal conductivity. A continuum is said to be *homogeneous* if its conductivity does not vary from point to point within the continuum, and *heterogeneous* if there is such variation. Furthermore, continua in which the conductivity is the same in all directions are said to be *isotropic*, whereas those in which there exists directional variation of conductivity are said to be *anisotropic.** Some materials consisting of a fibrous structure exhibit anisotropic character, for example, wood and asbestos. Materials having a porous structure, such as wool or cork, are examples of heterogeneous continua. In this text, except where explicitly stated otherwise, we shall be studying only the problems of isotropic continua. Because of the symmetry in the conduction of heat in isotropic continua, the flux of heat at a point must be normal to the *isothermal surface*† through this point.

Suppose now that the plate of Fig. 2–8 is isotropic but heterogeneous. Let the temperatures of two isothermal surfaces corresponding to the locations x and $x + \Delta x$ be T and $T + \Delta T$, respectively (Fig. 2–9). Since this plate may be assumed to be locally homogeneous, Eq. (2–71) can be used for a layer of the plate having the thickness Δx as $\Delta x \to 0$. Thus it becomes possible to state the differential form of Fourier's law of conduction, giving the heat flux at x in the direction of increasing x, as follows:

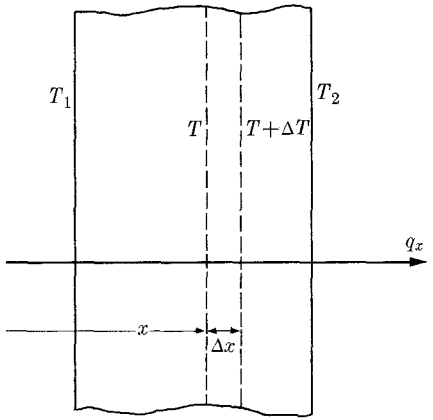


FIG. 2–9

$$q_x = -k \lim_{\Delta x \to 0} \left(\frac{\Delta T}{\Delta x} \right) = -k \frac{\partial T}{\partial x}, \tag{2–72}$$

(*Fourier's law for heterogeneous isotropic continua*). In Eq. (2–72), by introducing a minus sign we have made q_x positive in the direction of increasing x. It is important to note that this equation is independent of the temperature distribution. Thus, for example, in Fig. 2–10(a) $\partial T/\partial x < 0$ and $q_x > 0$, whereas in Fig. 2–10(b) $\partial T/\partial x > 0$ and $q_x < 0$. Both results agree with the second law of thermodynamics in that the heat diffuses from higher to lower temperatures.

---

* Once we have classified continua according to their conductivity, it becomes clear that the solids used in the experiments which suggest Fourier's law of conduction must necessarily be homogeneous and isotropic.

† A surface described instantaneously in a continuum such that at every point upon it the temperature is the same.

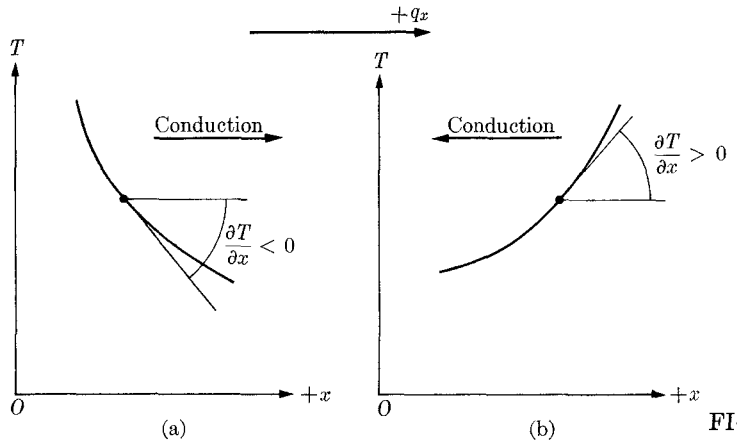


FIG. 2–10

Equation (2–72) may be readily extended to any isothermal surface if we state that the heat flux across an isothermal surface is

$$q_n = -k\frac{\partial T}{\partial n}, \tag{2–73}$$

where $\partial/\partial n$ represents differentiation along the normal to the surface. Denoting this flux by a vector $\mathbf{q}$ coinciding with n, we have

$$q_n = |\mathbf{q}|. \tag{2–74}$$

Here $\mathbf{q}$ may be expressed in terms of a coordinate system.* This yields

$$\mathbf{q} = -k\,\nabla T, \tag{2–75}$$

which is the *vectorial form of Fourier's law for heterogeneous isotropic continua.*

The heat flux at a point P across any nonisothermal surface is now determined by the heat flux across the isothermal surface through the same point (Fig. 2–11). If at P the normal vector $\mathbf{m}$ to a nonisothermal surface has direction cosines (α, β, γ) relative to a coordinate system, the magnitude of the heat flux across this surface is

$$q_m = \mathbf{q} \cdot \mathbf{m}. \tag{2–76}$$

---

* In terms of cartesian coordinates, for example,

$$\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors in the x-, y-, and z-directions, respectively. Noting from Eq. (2–72) that $q_x = -k(\partial T/\partial x)$, and similarly that $q_y = -k(\partial T/\partial y)$, $q_z = -k(\partial T/\partial z)$, we have

$$\mathbf{q} = -k\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)T,$$

which, by definition, is identical to Eq. (2–75).

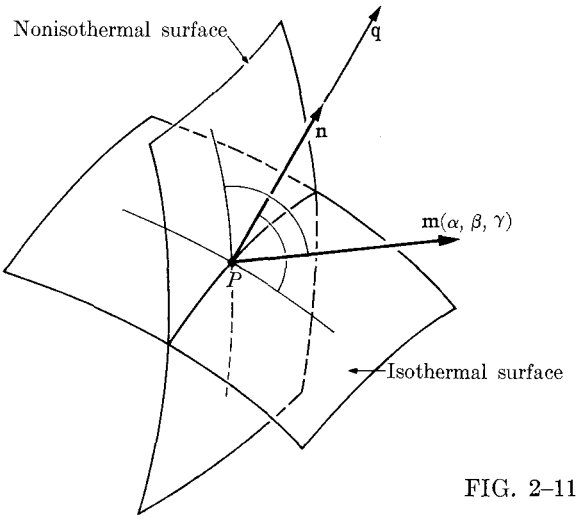


FIG. 2–11

Introducing Eq. (2–75) into Eq. (2–76) yields

$$q_m = -k(\nabla T \cdot \mathbf{m}). \tag{2–77}$$

Since, according to vector calculus,

$$\frac{\partial T}{\partial m} = \nabla T \cdot \mathbf{m},$$

Eq. (2–77) may also be written in the form

$$q_m = -k \frac{\partial T}{\partial m}, \tag{2–78}$$

which gives the magnitude of the heat flux across any surface; here $\partial/\partial m$ represents differentiation in the direction of the normal.

Thus far we have considered Fourier's law for isotropic continua only. In practice, anisotropic continua are also important. The most frequent examples of these are crystals, wood, and laminated materials such as transformer cores. For such continua the direction of the heat flux vector at a point is no longer normal to the isothermal surface through the point. Generalizing Fourier's law for isotropic continua, we may assume each component of the heat flux vector to be linearly dependent on all components of the temperature gradient at the point. Thus, for example, the cartesian form of *Fourier's law* for heterogeneous*

* The vectorial form of this law is

$$\mathbf{q} = -\boldsymbol{\kappa} \cdot \nabla T,$$

where $\boldsymbol{\kappa}$ is the *conductivity tensor;* the components of this tensor are called the *conductivity coefficients.*

anisotropic continua becomes

$$q_x = -\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y} + k_{13}\frac{\partial T}{\partial z}\right),$$
$$q_y = -\left(k_{21}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y} + k_{23}\frac{\partial T}{\partial z}\right), \tag{2–79}$$
$$q_z = -\left(k_{31}\frac{\partial T}{\partial x} + k_{32}\frac{\partial T}{\partial y} + k_{33}\frac{\partial T}{\partial z}\right).$$

The physical dimensions of the property k in British thermal units are $[k] \equiv$ Btu/ft·hr·°F. The numerical value of k for different continua varies from practically zero for gases under extremely low pressures to about 7000 Btu/ft·hr·°F for a natural copper crystal at very low temperatures. The value of k for a continuum depends in general on the chemical composition, the physical state, and the structure, temperature, and pressure.

In solids the pressure dependency, being very small, is always neglected. For narrow temperature intervals the temperature dependency may also be negligible. Otherwise a linear relation is assumed in the form

$$k = k_0(1 + \beta T), \tag{2–80}$$

where β is small and negative for most solids.

To illustrate the numerical values, the thermal conductivities of some gases, liquids, and solids are given in Fig. 2–12* as functions of the temperature. The experimental methods for determining the thermal conductivity of continua are many and varied. These, however, have been treated quite extensively in the literature, and will not be given in this text.†

Stefan-Boltzmann's law of radiation. Before the statement of this law is given, a brief review of a number of concepts seems appropriate.

From the viewpoint of electromagnetics, radiant heat transfer, like radio waves, light, cosmic rays, etc., is energy in the form of electromagnetic waves differing only in wavelength from other radiations. When radiant energy impinges on a surface, one fraction of it, α, is absorbed; another fraction, ρ, is reflected; and the remainder, τ, is transmitted. Thus

$$\alpha + \rho + \tau = 1, \tag{2–81}$$

where α, ρ, and τ are respectively called the *absorptivity*, *reflectivity*, and *transmissivity* of the surface. Equation (2–81) reduces to

$$\alpha + \rho = 1 \tag{2–82}$$

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* From M. Jakob and G. A. Hawkins, *Elements of Heat Transfer*. New York: John Wiley & Sons, 1957. Reproduced by permission.

† See, for example, Chapter 9 of Reference 14.

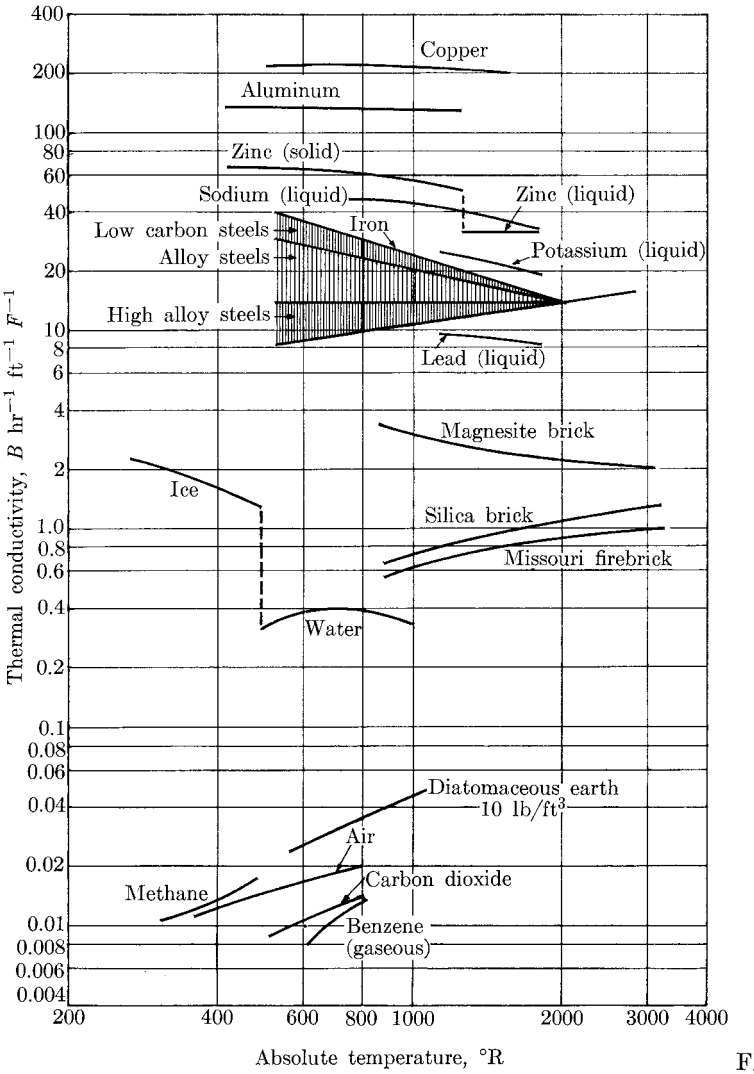


FIG. 2–12

for *opaque* continua, since $\tau = 0$, and to

$$\alpha + \tau = 1 \tag{2–83}$$

for *transparent* continua, since $\rho = 0$.

A surface which absorbs all radiation incident upon it ($\alpha = 1$) or at a specified temperature emits the maximum possible radiation is called a *black surface.*

The emissivity of a surface is defined as

$$\epsilon = q/q_b, \tag{2–84}$$

where q and q_b are the radiant heat fluxes from this surface and from a black surface, respectively, at the same temperature. Thus $\epsilon = 1$ for a black surface.

Under thermal equilibrium $\alpha = \epsilon$ for any surface.*

Consider now two isothermal surfaces A_1 and A_2 having the emissivities ϵ_1, ϵ_2 and the *absolute* temperatures T_1, T_2. These surfaces, together with a third insulated surface, complete an enclosure (Fig. 2–13). It has been shown experimentally by Stefan and later proved thermodynamically by Boltzmann that under steady conditions and in the presence of a nonabsorbing continuum or vacuum, the radiant heat flux q_{12} between the surfaces A_1 and A_2 is governed by *Stefan-Boltzmann's law of radiation* as follows:

$$q_{12} = \sigma \bar{\mathfrak{F}}_{12}(T_1^4 - T_2^4), \tag{2–85}$$

where σ is the *Stefan-Boltzmann constant*

$$0.17 \times 10^{-8} \text{ Btu/ft}^2\text{·hr·°R}^4.$$

$\bar{\mathfrak{F}}_{12}$ is a factor which, depending on the emissivity and the relative position of two surfaces, has the form

$$\frac{1}{\bar{\mathfrak{F}}_{12}} = \left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{\bar{F}_{12}} + \frac{A_1}{A_2}\left(\frac{1}{\epsilon_2} - 1\right), \tag{2–86}$$

where

$$\bar{F}_{12} = \frac{A_2 - A_1 F_{12}^2}{A_1 + A_2 - 2A_1 F_{12}}. \tag{2–87}$$

Here F_{12} is the so-called *geometric view factor*. Physically F_{12} represents the fraction of the total radiation from the surface A_1 which is intercepted by the surface A_2. This factor becomes unity for a surface which is enclosed by another surface or for two parallel plates having negligible radiation losses from the ends. As the insulated surface approaches zero, $\bar{F}_{12} \to F_{12}$.

For configurations including more than three surfaces, the evaluation of the radiant heat flux becomes involved. Furthermore, the determination of the geometric view factors for any but simple geometrics are often complex; hence these will not be given here.

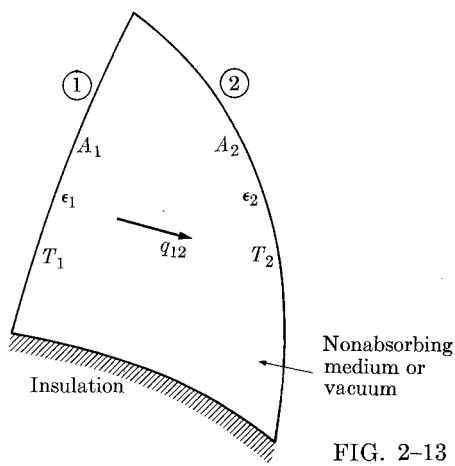


FIG. 2–13

* A result of Kirchhoff's law. See, for example, Reference 14, Section 4–2.

2–7. Equation of Conduction. Entropy Generation Due to Conductive Resistance

When we introduce Fourier's law, Eqs. (2–75) or (2–79), into the law of conservation of thermal energy, Eq. (2–64), the differential form of the equation of heat conduction may be obtained in terms of the temperature alone.

First let us consider an isotropic continuum. Inserting Eq. (2–75) into Eq. (2–64), we have the *equation of conduction for heterogeneous isotropic solids and frictionless incompressible fluids:*

$$\rho c \frac{dT}{dt} = \nabla \cdot (k \nabla T) + u''' . \tag{2–88}$$

Equation (2–88) may be rearranged by means of Eq. (2–44) to give

$$\rho c \frac{dT}{dt} = \nabla k \cdot \nabla T + k \nabla^2 T + u''', \tag{2–89}$$

where ∇^2 denotes the well-known *Laplacian operator.* If k is a function of space only, Eq. (2–89) is linear. On the other hand, when k depends on temperature alone, by use of the vectorial identity $\nabla k = (dk/dT)\nabla T$ Eq. (2–89) may be modified as

$$\rho c \frac{dT}{dt} = \frac{dk}{dT} (\nabla T)^2 + k \nabla^2 T + u''', \tag{2–90}$$

which is nonlinear. [Which term makes Eq. (2–90) nonlinear?] For homogeneous isotropic continua k is constant, and Eq. (2–89) reduces to the *equation of conduction for homogeneous isotropic solids and frictionless incompressible fluids:*

$$\frac{dT}{dt} = a \nabla^2 T + \frac{u'''}{\rho c}, \tag{2–91}$$

where

$$a = k/\rho c \tag{2–92}$$

is the so-called *thermal diffusivity.*

Next suppose that the continuum is anisotropic.* In terms of cartesian coordinates, for example, introducing Eq. (2–79) into Eq. (2–64) gives

$$\rho c \frac{dT}{dt} = \frac{\partial}{\partial x}\left(k_{11} \frac{\partial T}{\partial x} + k_{12} \frac{\partial T}{\partial y} + k_{13} \frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial y}\left(k_{21} \frac{\partial T}{\partial x} + k_{22} \frac{\partial T}{\partial y} + k_{23} \frac{\partial T}{\partial z}\right)$$
$$+ \frac{\partial}{\partial z}\left(k_{31} \frac{\partial T}{\partial x} + k_{32} \frac{\partial T}{\partial y} + k_{33} \frac{\partial T}{\partial z}\right) + u''', \tag{2–93}$$

* This case does not have any physical significance for fluids.

the *equation of conduction for heterogeneous anisotropic solids.** If the conductivity coefficients, though different from each other, remain constant in space, Eq. (2–93) reduces to

$$\rho c \frac{dT}{dt} = k_{11} \frac{\partial^2 T}{\partial x^2} + k_{22} \frac{\partial^2 T}{\partial y^2} + k_{33} \frac{\partial^2 T}{\partial z^2} + (k_{12} + k_{21}) \frac{\partial^2 T}{\partial x \, \partial y} + (k_{23} + k_{32}) \frac{\partial^2 T}{\partial y \, \partial z} + (k_{31} + k_{13}) \frac{\partial^2 T}{\partial z \, \partial x} + u''', \tag{2–94}$$

the *equation of conduction for homogeneous anisotropic solids.* The intended scope of this text prevents any further discussion of anisotropic continua.†

Once the temperature variation of any continuum undergoing a desired process is obtained, the entropy generation of this process, related to the temperature by the use of Fourier's law, may be readily evaluated. Hence in terms of isotropic continua, for example, introducing Eq. (2–75) into Eq. (2–69), we have

$$s''' = k \frac{(\nabla T)^2}{T^2} + \frac{u'''}{T}. \tag{2–95}$$

So far we have been able to derive expressions, namely, the equations of conduction given by Eqs. (2–88), (2–91), (2–93), and (2–94), which satisfy a partial differential equation in terms of the unknown temperature, rather than an algebraic equation. Since the solution of a differential equation involves a number of integration constants, the completion of the formulation requires that we state an equal number of appropriate conditions in space and time to determine these constants. This is the concern of the next section.

---

* In terms of the conductivity tensor $\boldsymbol{\kappa}$ the general representation of Eq. (2–93) is

$$\rho c \frac{dT}{dt} = \nabla \cdot (\boldsymbol{\kappa} \cdot \nabla T) + u''',$$

which, by use of the tensor calculus, may be rearranged in the form

$$\rho c \frac{dT}{dt} = (\nabla \cdot \boldsymbol{\kappa}) \cdot (\nabla T) + \boldsymbol{\kappa} : \nabla(\nabla T) + u'''.$$

Similarly, the general form of Eq. (2–94) is

$$\rho c \frac{dT}{dt} = \boldsymbol{\kappa} : \nabla(\nabla T) + u'''.$$

† Interested readers may refer to Sections 1–17, 1–18, 1–19, and 1–20 of Reference 2, and the literature cited in the same reference.

2–8. Initial and Boundary Conditions

These conditions are the mathematical descriptions of experimental observations. *Their number in the direction of each independent variable of a problem is equal to the order of the highest derivative of the governing differential equation in the same direction.* An example taken from conduction may illustrate this statement. Consider the equation of conduction written, for example, in cartesian coordinates for a homogeneous isotropic solid moving with velocity **V**,

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{u'''}{\rho c}, \qquad (2\text{–}96)$$

where v_x, v_y, v_z are the components of **V**. The solution of Eq. (2–96), regardless of the mathematical method employed, requires a single integration in time and a double integration in each of the three space variables involved. Thus, referring to the condition in time as the *initial condition* and the conditions in space as the *boundary conditions*, we say that Eq. (2–96), together with one initial and six boundary conditions, completes the differential formulation of the problem.* Let us now consider in detail the initial and boundary conditions appropriate for heat conduction problems.

Initial (volume) condition. For an unsteady problem the temperature of a continuum under consideration must be known at some instant of time. In many cases this instant is most conveniently taken to be the beginning of the problem. Mathematically speaking, if the initial condition is given by $T_0(\mathbf{r})$, the solution of this problem, $T(\mathbf{r}, t)$, must be such that at all points of the continuum

$$\lim_{t \to 0} T(\mathbf{r}, t) = T_0(\mathbf{r}). \qquad (2\text{–}97)$$

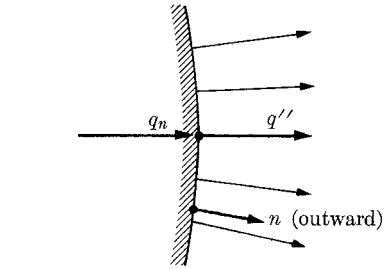
Boundary (surface) conditions. The most frequently encountered boundary conditions in conduction are as follows.

(1) *Prescribed temperature.* The surface temperature of the boundaries is specified to be a constant or a function of space and/or time. This is the easiest boundary condition from the viewpoint of mathematics, yet a difficult one to materialize physically, except for the limiting case $h \to \infty$ described below, under (4).

(2) *Prescribed heat flux.* The heat flux across the boundaries is specified to be a constant or a function of space and/or time. The mathematical description of this condition may be given in the light of Kirchhoff's current law; that is, the algebraic sum of heat fluxes at a boundary must be equal to zero. Hereafter the sign is to be assumed positive for the heat flux to the boundary and negative for that from the boundary. Thus, remembering that the statement

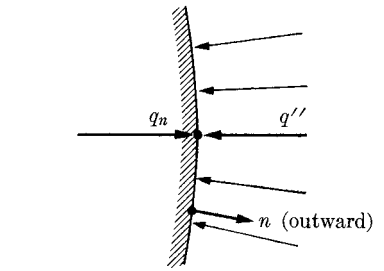
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* Clearly, the velocity involved in Eq. (2–96), although affecting the solution of this equation, does not change the number of initial and boundary conditions needed.



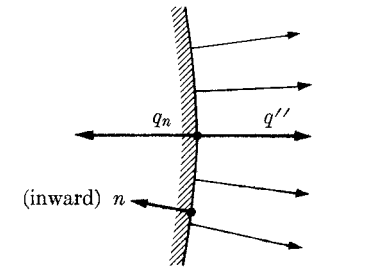
$$q_n - q'' = 0,$$
$$-k\left(\frac{\partial T}{\partial n}\right)_\sigma - q'' = 0.$$

(a)



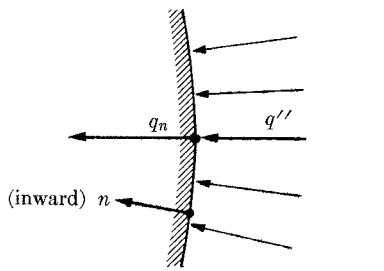
$$q_n + q'' = 0,$$
$$-k\left(\frac{\partial T}{\partial n}\right)_\sigma + q'' = 0.$$

(b)



$$-q_n - q'' = 0,$$
$$-\left[-k\left(\frac{\partial T}{\partial n}\right)_\sigma\right] - q'' = 0.$$

(c)



$$-q_n + q'' = 0,$$
$$-\left[-k\left(\frac{\partial T}{\partial n}\right)_\sigma\right] + q'' = 0.$$

(d)

FIG. 2–14

of Fourier's law, $q_n = -k(\partial T/\partial n)$, is independent of the actual temperature distribution, and selecting the direction of q_n conveniently such that it becomes positive, we have from Fig. 2–14

$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma = \pm q'', \tag{2–98}$$

where $\partial/\partial n$ denotes differentiation along the normal of the boundary. The

plus and minus signs of the left-hand side of Eq. (2–98) correspond to the differentiations along the inward and outward normals, respectively, and the plus and minus signs of the right-hand side correspond to the heat flux from and to the boundary, respectively.

A practical example of this case is encountered in the experimental evaluation of the forced-convection heat transfer coefficient in tubes as follows. Consider a constant internal energy as being generated electrically in the walls of an externally insulated tube through which a fluid flows in a prescribed manner. Under steady conditions and with the assumption that the electric resistivity and the thermal conductivity of the tube walls are constant, the fluid becomes subjected to a constant peripheral heat flux.*

(3) *No heat flux (insulation).* This, prescribed

$$\left(\frac{\partial T}{\partial n}\right)_{\sigma} = 0, \tag{2–99}$$

is a special form of the previous case, obtained by inserting $q'' = 0$ into Eq. (2–98). The illustrative example considered below indicates the practical importance of this boundary condition.

We wish to transfer heat from one surface of a flat electric heater plate through a solid plate, say plate 1, for a specific purpose (Fig. 2–15). Any heat transfer from the other surface of the heater is considered to be a heat loss, and is not desired. In practice the heat loss is reduced by placing another flat plate, plate 2 (insulator), next to the second surface of the heater. We are interested in the geometric and thermal properties of the insulator.

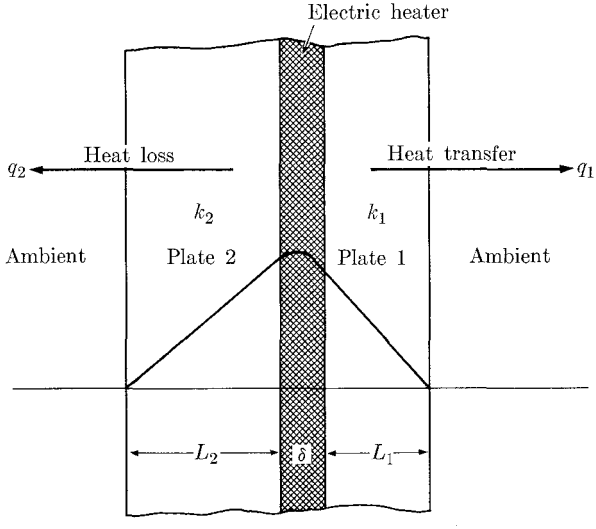


FIG. 2–15

* The same physical situation is reconsidered in (4). See also Problem 3–8.

Let us assume, for the sake of simplicity, that the left and right ambients have negligible thermal resistances.* Denoting the thickness of the heater by δ, the thermal conductivity and the thickness of the plates by k_1, k_2 and L_1, L_2, and the rate of internal energy generation per unit volume by u''', we have under steady conditions

$$\delta u''' = q_1 + q_2, \qquad q_1 \frac{L_1}{k_1} = q_2 \frac{L_2}{k_2},$$

and from these,

$$q_1 = \frac{\delta u'''}{1 + (L_1/k_1)/(L_2/k_2)}, \qquad q_2 = \frac{\delta u'''}{1 + (L_2/k_2)/(L_1/k_1)}.$$

The desired condition that $q_2 \to 0$ may be readily obtained by letting $(L_2/k_2)/(L_1/k_1) \to \infty$, or

$$\frac{L_2}{k_2} \gg \frac{L_1}{k_1}. \tag{2–100}$$

We learn from Eq. (2–100) that only the thickness and the thermal conductivity of plates are important for the conduction of heat through these plates.† Hence the heat loss through plate 2 can be eliminated by letting either $L_2 \to \infty$ or $k_2 \to 0$ compared with L_1 and k_1, respectively. Since a plate of $L_2 = \infty$ or $k_2 = 0$ is physically impossible, the foregoing insulation may never be accomplished in the absolute sense. The larger the thickness or the smaller the thermal conductivity, however, the better the insulation will be.

If the heat loss through plate 2 is to be completely eliminated, the use of another heater becomes necessary (Fig. 2–16). Then, by properly adjusting the

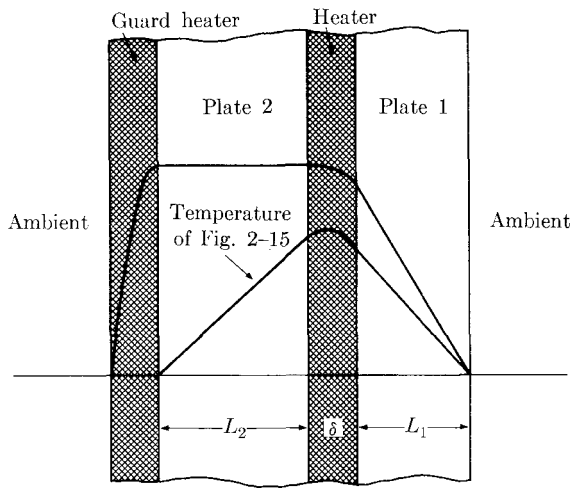
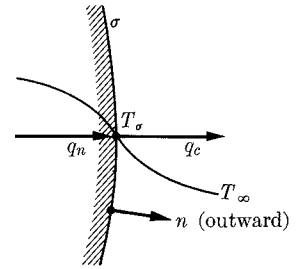


FIG. 2–16

* The case of finite resistance of the ambient is introduced in (4).

† The temperature drop (or temperature gradient) across a plate is immaterial for the conductive character of the plate.

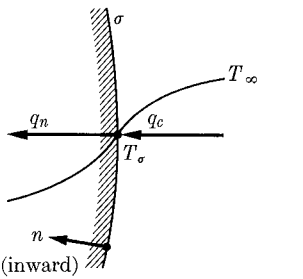
power supply to the second heater, all internal energy generated in the original heater may be transferred through plate 1. The second heater, often referred to as the *guard heater*, is an important experimental tool for the control of heat transfer, since it permits accurate thermal conductivity measurements.



$$q_n - q_c = 0,$$

$$-k\left(\frac{\partial T}{\partial n}\right)_\sigma - h(T_\sigma - T_\infty) = 0.$$

(a)



$$-q_n + q_c = 0,$$

$$-\left[-k\left(\frac{\partial T}{\partial n}\right)_\sigma\right] + h(T_\infty - T_\sigma) = 0.$$

(b)

FIG. 2–17

(4) *Heat transfer to the ambient by convection.* When the heat transfer across the boundaries of a continuum cannot be prescribed, it may be assumed to be proportional to the temperature difference between the boundaries and the ambient. Thus we have

$$q_c = h(T_\sigma - T_\infty), \tag{2–101}$$

where T_σ is the temperature of the solid boundaries, T_∞ is the temperature of the ambient at a distance far from the boundaries, and h, the proportionality constant,* is the so-called *heat transfer coefficient.* Equation (2–101) is *Newton's cooling law.* It is important, however, to note that this relation is not phenomenological like Fourier's law of conduction and Stefan-Boltzmann's law of radiation. Since it is based on an assumption only, it cannot be considered a natural (particular) law; it will therefore be referred to as the definition of the heat transfer coefficient. Despite its weak foundation, Eq. (2–101), being the only relation available for expressing the unspecified heat transfer to the ambient, plays a significant role in conduction problems.

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* h, by definition, is assumed to be positive.

TABLE 2–2

Condition		h (Btu/ft^2 · hr · °F)
Free convection	Gases	1–5
	Water	20–150
Forced convection	Gases	2–50
	Water	50–2000
	Viscous oils	10–300
	Liquid metals	1000–20,000
Phase change	Boiling liquids	500–10,000
	Condensing vapors	1000–20,000

Thus with the consideration that the sum of heat fluxes at the boundary must be equal to zero, and in the light of Eqs. (2–73) and (2–101), the required boundary condition may be stated in the form

$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma = h(T_\sigma - T_\infty), \tag{2–102}$$

where $\partial/\partial n$ denotes the differentiation along the normal. The plus and minus signs of the left member of Eq. (2–102) correspond to the differentiations along the inward and outward normals, respectively (Fig. 2–17). It should be kept in mind that q_n shown in Fig. 2–17 is a positive quantity, obtained by arbitrarily selecting it in the direction of the normal. Actually, Eq. (2–102) is independent of the temperature distribution and the direction of the heat transfer.

The range of values of heat transfer coefficients occurring under various conditions will now be presented, to give the reader a feeling for the order of magnitudes involved. It should be remembered that h, similar to but more strongly than k, depends on certain variables. These may include the space, time, geometry, flow conditions, and physical properties. The spacewise averaged, steady values of commonly encountered heat transfer coefficients are given in Table 2–2.

The wide variation in the values of heat transfer coefficients suggests further investigations of the boundary condition under study for the limiting values of h. This will be done in terms of a frequently encountered practical situation as follows. Consider a tube of inner and outer radii R_i, R_o through which a fluid flows under specified steady conditions (Fig. 2–18). The steady and uniform internal energy per unit volume is generated at the rate of u''' in the tube walls. The temperature of the surroundings and the bulk temperature of the

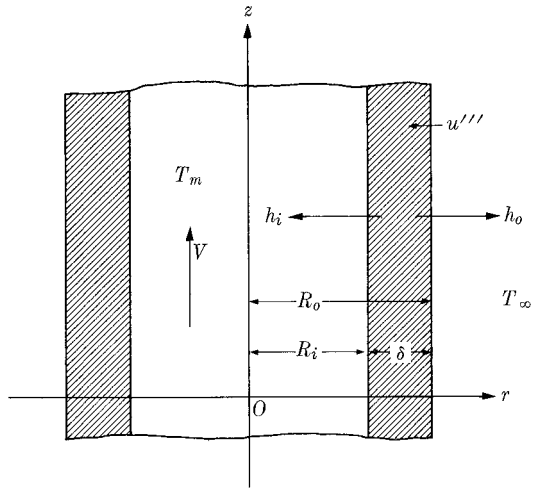


FIG. 2–18

fluid* are T_∞ and T_m, respectively, and the inside and outside heat transfer coefficients are h_i, h_o. The radial boundary conditions for the tube may be written in the form

$$q_i = +k\frac{\partial T(R_i, z)}{\partial r} = h_i[T(R_i, z) - T_m],$$

$$q_o = -k\frac{\partial T(R_o, z)}{\partial r} = h_o[T(R_o, z) - T_\infty].$$

Let us first consider the case of an insulated tube, where $q_o = 0$. In this case, under steady conditions all the internal energy generated in the tube walls is transferred to the inside fluid. Thus $q_i = u'''\delta$. Since q_i is constant for a given u''' and δ, the larger the inside heat transfer coefficient h_i gets, the smaller the temperature difference $T(R_i, z) - T_m$ becomes. In the limit as $h_i \to \infty$, $T(R_i, z)$ tends to T_m, so that the present boundary condition is reduced to that of prescribed surface temperature given under (1). The boiling of liquids in insulated tubes is an example of this case. For a constant q_i, the smallest h_i gives the largest temperature difference between $T(R_i, z)$ and T_m.

Next consider the case of a bare tube. Now $u'''\delta$ is transferred to the surroundings as well as to the inside fluid. Assume that T_m and T_∞ are of the same order, and that the tube wall is thin enough that the difference between

* The bulk temperature of a fluid is

$$T_m = \frac{1}{\pi R_i^2 \rho_m c_{p_m} V}\int_0^{R_i} 2\pi r v(r)\rho c_p T(r)\, dr,$$

where ρ_m, c_{p_m}, and V are the density, specific heat at constant pressure, and mean velocity of the fluid, all evaluated at the bulk temperature T_m.

$T(R_i, z)$ and $T(R_o, z)$ may be neglected. Then the heat transfer to the inside fluid and to the surroundings becomes approximately proportional to the inside and outside heat transfer coefficients, respectively. If, for example, $h_o \ll h_i$, the heat transfer to the surroundings may be neglected compared with that to the inside fluid. Thus the outer surface of the tube may be assumed to be insulated, the boundary condition described under (3). An example of this case is water flowing through a tube (forced convection to liquids) surrounded by the stationary atmosphere (free convection to gases). The temperature sketches for three cases, $h_i \sim h_o$, $h_o \ll h_i$, either one of these and large h_i, are left to the reader. From the foregoing discussion we learn the important fact that *the magnitude of the heat transfer coefficient is decisive for the type of boundary condition to be used in the formulation of a problem.*

The study of the magnitude of h suggests a similar investigation in regard to the magnitude of k. For this purpose let us return to the case of an insulated tube. For a given q_i, the larger the thermal conductivity, the smaller the temperature gradient $\partial T(R_i, z)/\partial r$. In the limit as $k \to \infty$, $\partial T(R_i, z)/\partial r$ approaches zero, which implies that the radial temperature distribution in the tube wall can be neglected. This, since it leads to a radially *lumped analysis*, may considerably simplify the formulation of the problem. On the other hand, small or moderate values of k require a radially *distributed analysis*. (Lumped and distributed analyses will be considered in the formulation of the five illustrative examples given in Section 2–10.) Thus *the magnitude of the thermal conductivity plays an important role in the formulation of the conduction equation of a problem.*

Finally, the dimensionless form of the boundary condition under consideration, written in the form

$$\pm\left[\frac{\partial T}{\partial(r/R)}\right]_\sigma = \frac{hR}{k}(T_\sigma - T_\infty), \tag{2–103}$$

indicates that the simultaneous effects of h and k may be investigated in terms of a single dimensionless number, the *Biot modulus*,

$$hR/k = \mathrm{Bi},$$

where R is a characteristic length. Rearranged in the form of $(R/k)/(1/h)$, the Biot modulus may physically be interpreted as the ratio of the internal and external resistances of a problem in the direction for which Eq. (2–103) applies.

(5) *Heat transfer to the ambient by radiation.* Let us reconsider Fig. 2–13, and find, for example, the boundary condition prescribing heat transfer by radiation from the boundaries of continuum 1.

When T_1 is uniform but unspecified, to express the heat flux across the surfaces of 1 by conduction and radiation the required boundary condition may be written in the form

$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma = \sigma\bar{\mathfrak{F}}_{12}(T_1^4 - T_2^4), \tag{2–104}$$

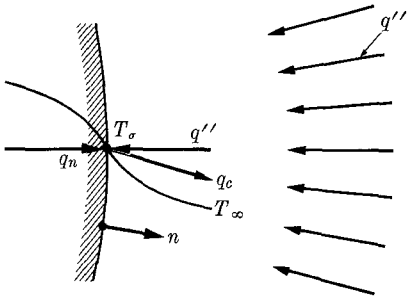


FIG. 2–19

where, as before, the plus and minus signs of the conduction term correspond to differentiation in the direction of inward and outward normals, respectively. Equation (2–104) is independent of the actual temperature distribution. Since it involves the fourth power of the dependent variable, it is a nonlinear boundary condition.

The combination of Eqs. (2–102) and (2–104) gives the simultaneous heat transfer by convection and radiation from the boundaries of the continua. In practice, such simultaneous transfer is the actual case. The importance of radiation relative to convection depends, to a large extent, on the temperature level; radiation increases rapidly with increasing temperature. Even at room temperatures, however, for low rates of convection, say free convection to air, radiation may contribute up to fifty percent of the total heat transfer.

(6) *Prescribed heat flux acting at a distance.* Consider a continuum that transfers heat to the ambient by convection while receiving the net radiant heat flux q'' from a distant source (Fig. 2–19).* The heat transfer coefficient is h, and the ambient temperature T_∞.

This boundary condition may be readily obtained as

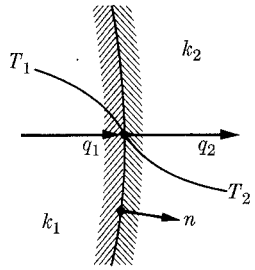
$$\pm k\left(\frac{\partial T}{\partial n}\right)_\sigma + q'' = h(T_\sigma - T_\infty), \tag{2–105}$$

where the signs of the conduction term depend on the direction of normal in the usual manner. Equation (2–105), like Eq. (2–104), is independent of the actual temperature distribution. Any body surrounded by the atmosphere, capable of receiving radiant heat, and near a radiant source (a light bulb or a sun lamp) or exposed to the sun exemplifies the foregoing boundary condition.

(7) *Interface of two continua of different conductivities k_1 and k_2.* When two continua have a common boundary (Fig. 2–20), the heat flux across this boundary evaluated from both continua, regardless of the direction of normal, gives

$$k_1\left(\frac{\partial T_1}{\partial n}\right)_\sigma = k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma. \tag{2–106}$$

* Hereafter figures illustrating the statement of boundary conditions are drawn for one direction of the boundary normal.



$$q_1 - q_2 = 0,$$

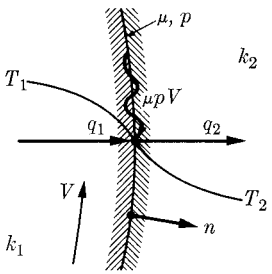
$$-k_1\left(\frac{\partial T_1}{\partial n}\right)_\sigma - \left[-k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma\right] = 0.$$

FIG. 2–20

Furthermore, a second condition may be specified along this boundary relating the temperatures of the two continua. If the continua are assumed, for example, to be solid and in intimate contact, mathematical idealization suggests the equality of temperatures

$$(T_1)_\sigma = (T_2)_\sigma. \tag{2–107}$$

However, Eq. (2–107) is a difficult condition to satisfy in practice. Even for perfectly smooth surfaces pressed together, heat transfer takes place between the two continua across the so-called *contact resistance.** This resistance, which is difficult to measure or specify, causes a temperature difference between the continua along the interface. Despite this fact, Eq. (2–107) necessarily finds extensive use in the formulation of conduction problems. Composite walls and insulated tubes are well-known examples of this case.



$$q_1 + \mu pV - q_2 = 0,$$

$$-k_1\left(\frac{\partial T_1}{\partial n}\right)_\sigma + \mu pV - \left[-k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma\right] = 0.$$

FIG. 2–21

(8) *Interface of two continua in relative motion.* Consider two solid continua in contact, one moving relative to the other (Fig. 2–21). The local pressure on the common boundary is p, the coefficient of dry friction μ, and the relative velocity V.

Noting that the heat transfer to both continua by conduction is equal to the work done by friction, we have

$$\pm k_1\left(\frac{\partial T}{\partial n}\right)_\sigma + \mu pV = \pm k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma, \tag{2–108}$$

where the minus signs of the conduction terms correspond to the normal shown

* See References 15 and 16.

in Fig. 2–21. Again, for idealized intimate contact we may assume that the temperatures of the two continua are the same on the boundary, as expressed previously by Eq. (2–107).

The friction brake is an important practical case of the foregoing boundary condition. However, wear and high temperatures make this boundary condition impractical for continuous operation. The obvious remedy, lubrication, is beyond the scope of the text and is not considered here.

(9) *Moving interface of two continua (change of phase).* When part of a continuum has temperatures below the temperature at which the continuum changes from one phase to another by virtue of the liberation or absorption of heat, there exists a moving boundary between the two phases. For problems in this category, the way in which the boundary moves has to be determined together with the temperature variation in the continuum.

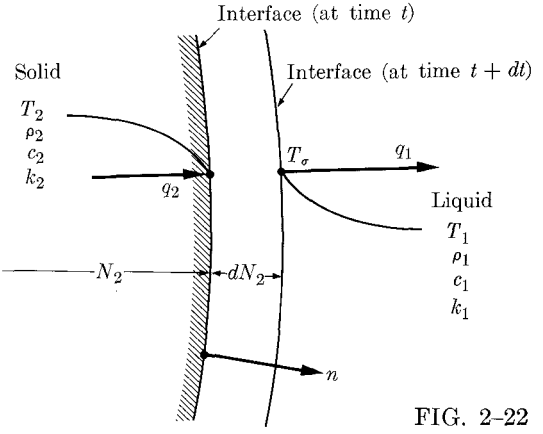


FIG. 2–22

Consider, for example, the solidification of a liquid. Here our concern is the boundary condition on the moving interface $N_2(t)$ (Fig. 2–22). The thermal properties of the liquid and solid are distinguished by the subscripts 1 and 2, respectively. Since the densities of the two phases are not the same, in the time interval dt the solid of thickness dN_2 is formed from the liquid of thickness dN_1. Applying the first law of thermodynamics to the system shown in Fig. 2–23, whose initial state is the liquid of thickness dN_1 and whose final state is the solid of thickness dN_2, we have

$$\rho_2 u_2\, dN_2 - \rho_1 u_1\, dN_1 = q_2\, dt - q_1\, dt - p(dN_2 - dN_1), \quad (2\text{–}109)$$

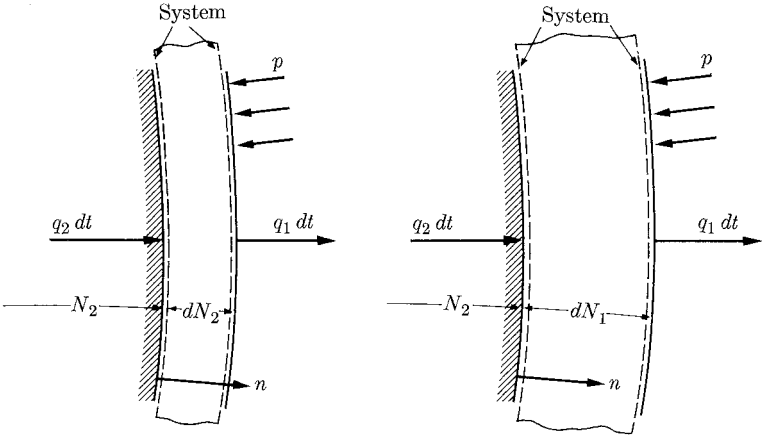
where p is the pressure of the continuum.

Noting the continuity

$$\rho_2\, dN_2 = \rho_1\, dN_1, \quad (2\text{–}110)$$

and the latent heat of fusion

$$h_{sl} = h_1 - h_2, \quad (2\text{–}111)$$



(2) Final state (solid) (1) Initial state (Liquid) FIG. 2–23

we may write the rate form of Eq. (2–109) as

$$-\rho_2 h_{sl} \frac{dN_2}{dt} = q_2 - q_1. \tag{2–112}$$

Expressing q_1 and q_2 by Fourier's law, we get finally

$$-\rho_2 h_{sl} \frac{dN_2}{dt} = \pm k_2 \left(\frac{\partial T_2}{\partial n}\right)_\sigma - \left[\pm k_1 \left(\frac{\partial T_1}{\partial n}\right)_\sigma\right], \tag{2–113}$$

where the plus and minus signs of the conduction terms correspond to the differentiation $\partial/\partial n$ along the inward and outward normals, respectively, of the solid phase.

The foregoing boundary condition may also be obtained by an alternative procedure* as follows. Since $\rho_2 \neq \rho_1$, say $\rho_2 > \rho_1$, the process of solidification gives rise to a velocity V_1 in the fluid proportional to the rate of the difference between the volumes of the two phases (Fig. 2–24a). Thus we have

$$V_1 = \frac{dN_1 - dN_2}{dt},$$

which may be rearranged by continuity, Eq. (2–110), as

$$V_1 = \left(\frac{\rho_2}{\rho_1} - 1\right) \frac{dN_2}{dt}. \tag{2–114}$$

To simplify the analysis, let us imagine that an observer is traveling with the

* A similar method is employed to evaluate the velocity of propagation of a plane pressure pulse in a stationary compressible fluid filling a pipe of uniform cross section. See, for example, Reference 5, Chapter 3, p. 44.

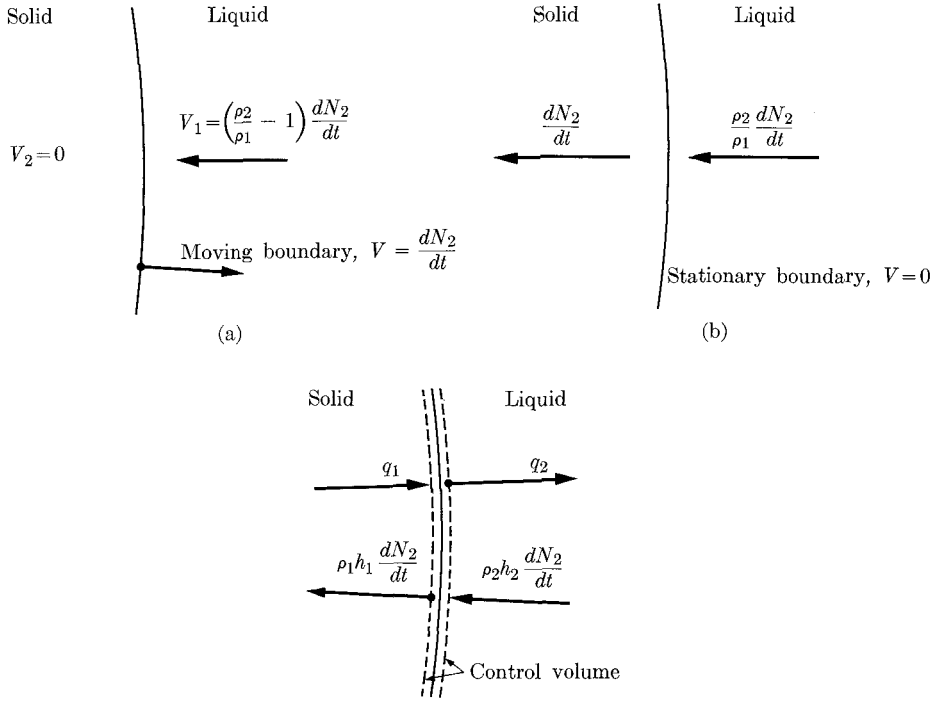


FIG. 2–24

moving boundary. Figure 2–24(b) shows the appearance of the solidification process to such an observer. Then the first law of thermodynamics applied to a control volume surrounding the stationary boundary of Fig. 2–24(c) yields

$$0 = \rho_2(h_2 - h_1)\frac{dN_2}{dt} + q_2 - q_1. \tag{2–115}$$

Given Eq. (2–111), Eq. (2–115) reduces to Eq. (2–112).

The problem of solidification may be simplified considerably when the temperature variation in the liquid is not of interest. In this case, if we express q_1 in terms of a heat transfer coefficient h, Eq. (2–113) may be rearranged as

$$-\rho_2 h_{sl}\frac{dN_2}{dt} = \pm k_2\left(\frac{\partial T_2}{\partial n}\right)_\sigma - h(T_\sigma - T_\infty), \tag{2–116}$$

where T_σ is the temperature of solidification and T_∞ the temperature of the liquid far from the moving boundary. Problems involving change of phase are of great practical importance. Ice formation both in geophysics and ice manufacturing, the solidification of metals in casting, and the condensation and evaporation of fluids are typical examples.

2–9. Methods of Formulation

In the preceding sections of this chapter we have established the general formulation of conduction phenomena. We might now expect to obtain the formulation of any specific problem from the general formulation. This, of course, is possible, but it is not always convenient, especially if the problem under consideration is to be lumped in one or more directions. (This point will be clarified by the problem of Fig. 2–27.) Furthermore, the application of the general formulation to a specific problem is a mathematical process which eliminates the physics of the formulation, an important aspect of practical problems. By contrast, the physical approach which will be stressed in this text treats each problem individually from the start of its formulation by bringing the physics into each phase of the formulation. To illustrate this statement let us compare the two methods in the light of three problems requiring the one-dimensional formulation of the first law of thermodynamics.

The first problem is that of the one-dimensional cartesian system shown in Fig. 2–25. When we equate the time rate of change of internal energy to the net heat transfer across the boundaries of the system, the physical approach yields

$$\rho \frac{\partial u}{\partial t} + \frac{\partial q_x}{\partial x} = 0. \qquad (2\text{–}117)$$

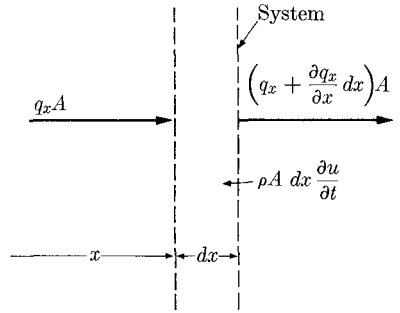


FIG. 2–25

The general formulation, reduced to the one-dimensional cartesian form of Eq. (2–57), gives the same result.

Next, let us consider an insulated solid rod of radius R, cross section A, and periphery P (Fig. 2–26). By either method, the first law of thermodynamics stated for the one-dimensional system shown in Fig. 2–26 yields the result of the previous problem, Eq. (2–117).

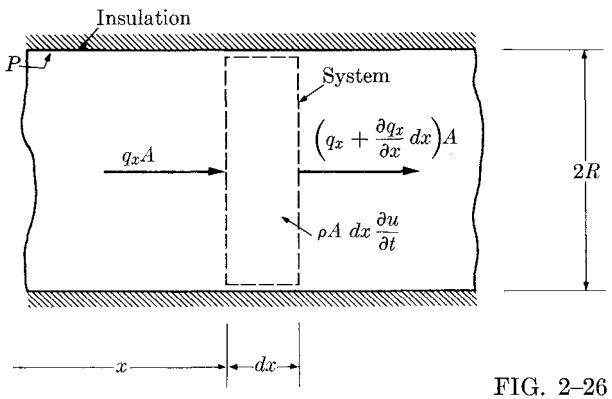


FIG. 2–26

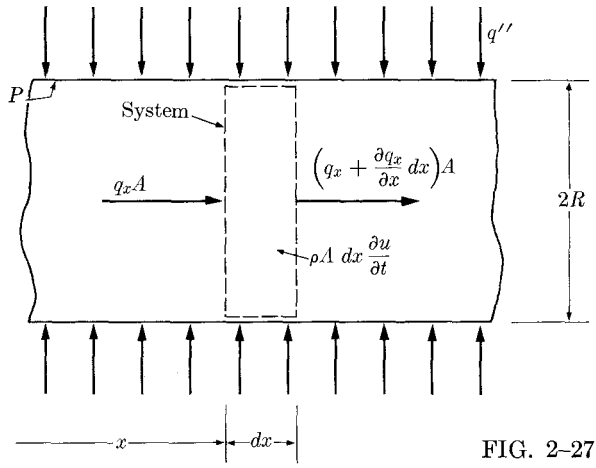


FIG. 2–27

Finally, consider the solid rod of the previous problem now subjected to the uniform peripheral heat flux q'' (Fig. 2–27). The physical approach, in which we apply the first law to the one-dimensional system shown in Fig. 2–27, results in

$$\rho \frac{\partial u}{\partial t} + \frac{\partial q_x}{\partial x} = \frac{q'' P}{A}. \tag{2–118}$$

By contrast, the one-dimensional form of the general formulation, leading again to Eq. (2–117), does not include the effect of the peripheral heat flux. This difficulty, however, may be circumvented by considering instead the two-dimensional form of Eq. (2–57),

$$\rho \frac{\partial u^*}{\partial t} + \frac{\partial q_x^*}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r q_r^*) = 0, \tag{2–119}$$

where u^*, q_x^*, and q_r^* now depend on r as well as x and t. Next, averaging Eq. (2–119) radially, that is, multiplying each term by $2\pi r\, dr$ and integrating the result over the interval $(0, R)$, yields

$$A\rho \frac{\partial u}{\partial t} + A \frac{\partial q_x}{\partial x} + 2\pi r q_r \Big|_0^R = 0, \tag{2–120}$$

which is identical to Eq. (2–118). Here the radially averaged value of a dependent variable, say u, is defined as

$$u(x, t) = \frac{2\pi}{A} \int_0^R r u^*(r, x, t)\, dr.$$

The discussion on the foregoing three examples may be generalized as follows. A given problem may be formulated either by considering the appropriate

specific case of the general formulation or by following, from the start, an individual formulation suitable to the problem. Whenever the general formulation is available the former method may be used, but this requires the mathematical interpretation of the general formulation in the light of the problem under consideration. The latter method, on the other hand, involves following certain steps in a basic procedure for individual formulation, given below. For one- or multidimensional problems which are formulated to include all dimensions of the problem (such as the first two of the foregoing examples), the general or mathematical approach proves to be slightly shorter than the individual or physical approach. However, for multidimensional problems which we wish to formulate in fewer dimensions, that is, which we wish to lump in one or more directions (such as the third example), the mathematical approach, requiring an averaging process, becomes lengthy and inconvenient.

The foregoing argument and the emphasis, in this text, on practical applications of the study of heat transfer thus suggest that the physical approach be the preferred method of formulation. For convenience and later reference, this method of formulation is summarized in the following five steps:

(i) *Define an appropriate system or control volume.* This step includes the selection of (a) a coordinate system, (b) a lumped or distributed formulation, and (c) a system or control volume in terms of (a) and (b).

(ii) *State the general laws for* (i). The general laws, except in their lumped forms, are written in terms of a coordinate system. The differential forms of these laws depend on the direction but not the origin of coordinates, whereas the integral forms depend on the origin as well as the direction of coordinates. Although the differential forms apply locally, the lumped and integral forms are stated for the entire system or control volume.

(iii) *State the particular laws for* (ii). The particular law describing the diffusion of heat (or momentum, mass, or electricity) is differential, applies locally, and depends on the direction but not the origin of coordinates.

(iv) *Obtain the governing equation from* (ii) *and* (iii). This, such as the equation of conduction, may be an algebraic, differential, or other equation involving the desired dependent variable, say the temperature, as the only unknown. The governing equation (except for its flow terms) is independent of the origin and direction of coordinates.

(v) *Specify the initial and/or boundary conditions pertinent to* (iv). These conditions depend on the origin as well as the direction of coordinates.

2–10. Examples

In this section the emphasis is placed on formulation; however, for those problems whose formulation leads to an ordinary differential equation of first order or to one of second order with constant coefficients we shall also give the solution.

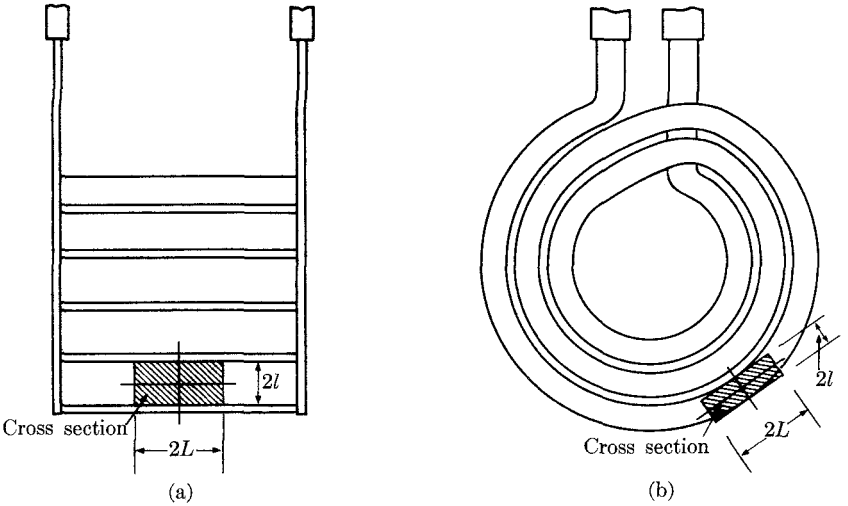


FIG. 2–28

Example 2–1. Consider an electric heater made from a solid rod of rectangular cross section ($2L \times 2l$) and designed according to one of the forms shown in Fig. 2–28. The temperature variation along the rod can be neglected. In addition, the effect of curvature is negligibly small for the coil type of heater of Fig. 2–28(b). The internal energy generation u''' in the heater is uniform. The heat transfer coefficient is denoted by h and the ambient temperature by T_∞. We wish to formulate the steady conduction problem suitable to this heater.

Proceeding according to the five basic steps mentioned in the previous section, we shall now give the lumped, differential, and integral formulations of the problem.

I. *Lumped formulation.*

(i) *System or control volume.* The lumped system covers the entire cross section of the heater (Fig. 2–29). Since the problem is assumed to be two-dimensional, the length of the rod(s) does not affect the formulation; for purposes of illustration, however, let us consider a unit length.

(ii) *General law.* The first law of thermodynamics, Eq. (2–16), applied to Fig. 2–29 yields

$$0 = -2q_1(2L \cdot 1) - 2q_2(2l \cdot 1) + u'''(2L \cdot 2l \cdot 1). \qquad (2\text{–}121)$$

(iii) *Particular law.* The formulation, being lumped, does not require any particular law.

(iv) *Governing equation.* The absence of a particular law makes the governing equation identical to the general law.

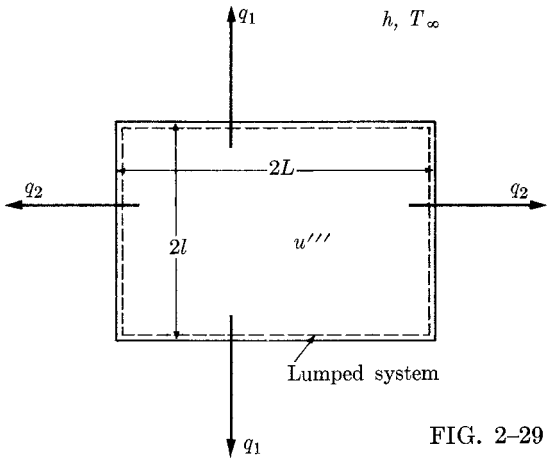


FIG. 2–29

(v) *Initial and boundary conditions.* The requirement of a steady formulation eliminates the need of any initial condition. The definition of h gives the single boundary condition

$$q_1 \text{ (and } q_2) = h(T - T_\infty). \tag{2–122}$$

Equations (2–121) and (2–122) complete the lumped formulation of the problem. If we introduce the latter into the former, this formulation may be written in terms of the unknown temperature T as follows:

$$0 = -2h(2L \cdot 1)(T - T_\infty) - 2h(2l \cdot 1)(T - T_\infty) + u'''(2L \cdot 2l \cdot 1). \tag{2–123}$$

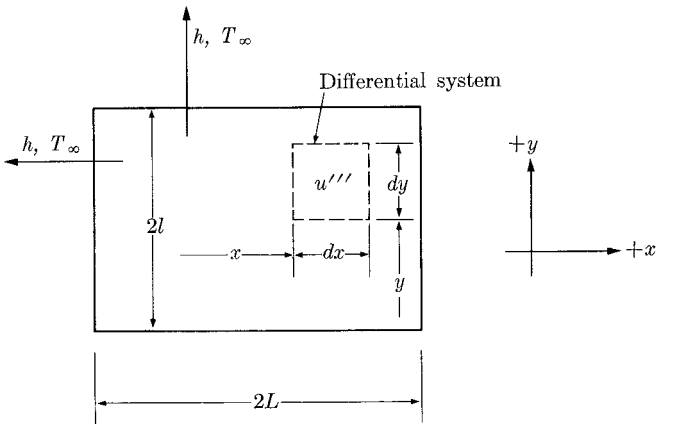
The simplicity of Eq. (2–123) readily allows the lumped temperature of the heater rod to be obtained as

$$T = T_\infty + \frac{u'''(2L \cdot 2l)}{2h(2L + 2l)}. \tag{2–124}$$

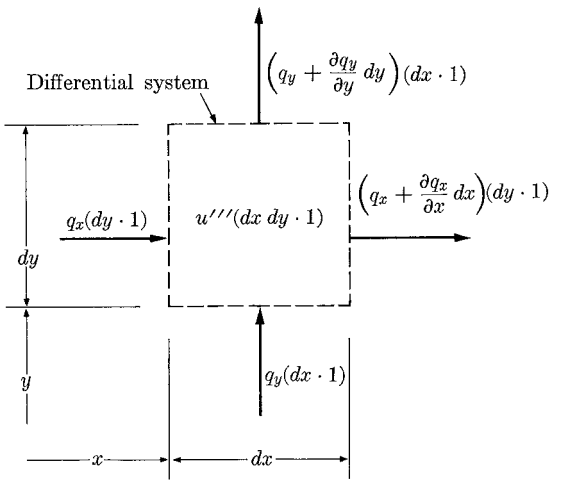
In the limit as $h \to \infty$ the temperature of the heater approaches the ambient temperature T_∞. [See the discussion in Section 2–8 regarding the boundary condition of type (4).]

II. *Differential formulation*

(i) *System or control volume.* Consider the two-dimensional differential system shown in Fig. 2–30. The horizontal direction is arbitrarily denoted by x and the vertical one by y. The direction and origin of the coordinates need not be specified yet. To fix ideas, however, we designate the rightward x and the upward y as positive.



(a)



(b)

FIG. 2–30

(ii) *General law.* The first law of thermodynamics applied to the differential system of Fig. 2–30(a) and interpreted in terms of Fig. 2–30(b) gives

$$0 = +q_x(dy \cdot 1) - \left(q_x + \frac{\partial q_x}{\partial x}\, dx\right)(dy \cdot 1)$$

$$+ q_y(dx \cdot 1) - \left(q_y + \frac{\partial q_y}{\partial y}\, dy\right)(dx \cdot 1) + u'''(dx \cdot dy \cdot 1),$$

which may be simplified to

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + u''' = 0. \tag{2–125}$$

(iii) *Particular law.* The two cartesian components of the vectorial form of Fourier's law to be used for isotropic continua are

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}. \tag{2–126}$$

(iv) *Governing equation.* Introducing Eq. (2–126) into Eq. (2–125) gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + u''' = 0, \tag{2–127}$$

which for constant k reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{u'''}{k} = 0. \tag{2–128}$$

Equation (2–127) or Eq. (2–128) is the governing equation (of conduction) for the problem under study. It is clear that these equations may readily be obtained from the general vectorial forms given by Eqs. (2–88) and (2–91) by considering their steady two-dimensional cartesian forms.

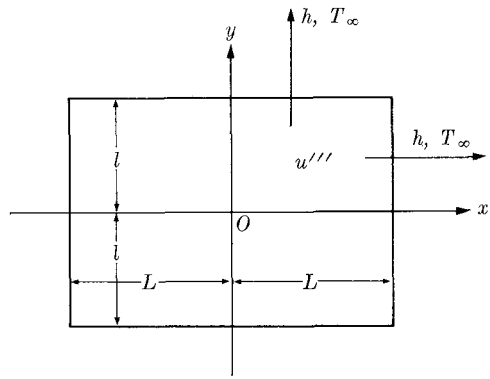


FIG. 2–31

(v) *Initial and boundary conditions.* As with the lumped formulation, no initial condition is needed. The order of highest x- and y-derivatives of Eq. (2–128), on the other hand, requires that two boundary conditions be specified in each direction. Before these can be stated, of course, the origin and direction of coordinates must be chosen. Noting the thermal as well as the geometric symmetry of the problem, we select the coordinate system shown in Fig. 2–31.

Thus the boundary conditions may be written in the form

$$\frac{\partial T(0, y)}{\partial x} = 0, \qquad -k\frac{\partial T(L, y)}{\partial x} = h[T(L, y) - T_\infty],$$
$$\frac{\partial T(x, 0)}{\partial y} = 0, \qquad -k\frac{\partial T(x, l)}{\partial y} = h[T(x, l) - T_\infty]. \tag{2–129}$$

[Restate these conditions in the coordinate system whose origin is at one of the corners of the heater. Defend Eq. (2–129) compared with the new form of the boundary conditions.]

Equation (2–127) or Eq. (2–128), together with Eq. (2–129), completes the differential formulation of the problem. The solution of this problem requires further mathematical background and is deferred to Chapter 4. (See Example 4–10, which deals with the limiting case $h \to \infty$.)

III. *Integral formulation*

(i) *System or control volume.* This formulation, as demonstrated below, requires the simultaneous use of the systems of the lumped and differential formulations (Fig. 2–32).

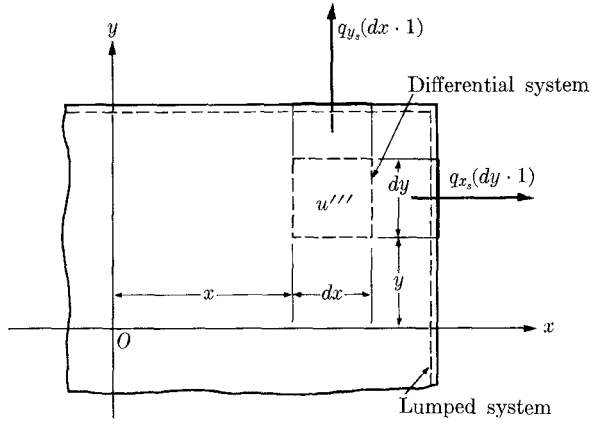


FIG. 2–32

(ii) *General law.* The first law of thermodynamics applied to the lumped system of Fig. 2–32, but with its terms interpreted by the differential system of the same figure, results in

$$-\int_0^l (q_x)_s\, dy - \int_0^L (q_y)_s\, dx + u'''Ll = 0. \tag{2–130}$$

The same result may also be obtained following the mathematical approach instead, by integrating the differential form of the first law of thermodynamics, Eq. (2–125), over the cross section of the heater. This yields

$$\int_0^L \int_0^l \left(-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + u'''\right) dx\, dy = 0. \tag{2–131}$$

The equality of Eqs. (2–130) and (2–131) may be readily shown by carrying out the appropriate integrations in Eq. (2–131).

(iii) *Particular law.* Since q_x and q_y apply locally, Fourier's law given by Eq. (2–126) is equally valid for the present case.

(iv) *Governing equation.* Introducing Eq. (2–126) into Eqs. (2–130) and (2–131) gives the integral form of the equation of conduction as

$$\int_0^l \left(\frac{\partial T}{\partial x}\right)_s dy + \int_0^L \left(\frac{\partial T}{\partial y}\right)_s dx + \frac{u'''}{k} Ll = 0 \tag{2–132}$$

or

$$\int_0^L \int_0^l \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{u'''}{k}\right) dx\, dy = 0. \tag{2–133}$$

Thus we have two integral forms corresponding to the differential formulation of the equation of conduction, one obtained by physical considerations, the other by the mathematical approach, which involves the integration of the suitable differential form over the cross section of the heater. It is clear that Eq. (2–133) is easier to establish than Eq. (2–132) when the corresponding differential form is available. The two equations are, of course, identical.

(v) *Initial and boundary conditions.* Since the terms of Eqs. (2–132) and (2–133) apply locally, the initial and boundary conditions of the differential formulation are also valid for the present integral formulation. Hence Eq. (2–132) or Eq. (2–133), together with Eq. (2–129), completes the integral formulation of the problem.

The integral formulation is useful for obtaining approximate solutions, which are convenient for problems whose exact solution is rather involved algebraically, and indispensable for complex problems having no exact solution. Solution of the integral formulation requires no further mathematics than that which we assume the reader to have; hence we shall give the method here. This method is based on the selection of an approximate profile for the unknown (dependent) variable, say the temperature. The profile, containing an unknown *parameter* to be determined, is assumed to be composed of the product of simple (polynomial, circular, etc.) functions.* Each function in this product depends on only one of the independent variables entering the problem, and is chosen such that the boundary conditions are satisfied.† When this product form is intro-

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* This, however, is an assumption only, and may not lead to a solution. The existence of a solution implies the validity of the assumption.

† Although not common in conduction heat transfer, the integral method is extensively used under the name of Karman-Polhausen procedure for approximate solutions of the velocity and temperature boundary-layer problems of fluid mechanics and convection heat transfer.

duced into the integral formulation, the result of integration specifies the unknown parameter,* and when, in turn, this value of the parameter is inserted into the product form, an approximate solution is obtained for the problem under consideration.

Let us now apply this general procedure to the present problem. For the temperature of the heater suppose that a product solution exists in the form

$$T(x, y) - T_\infty = X(x)Y(y),$$

where X and Y are functions of x and y, respectively. Restricting ourselves to the case of large h and assuming, for example, parabolic profiles in both directions such that they satisfy the boundary conditions, we may write a first approximation of the heater temperature as

$$T(x, y) - T_\infty = (L^2 - x^2)(l^2 - y^2)a_0, \tag{2–134}$$

where a_0 is the unknown parameter to be determined. Equation (2–134) is the first-order polynomial *Ritz profile*, from the well-known *Ritz method* of the variational calculus, which will be discussed in Chapter 8.

Inserting Eq. (2–134) into Eq. (2–132) or Eq. (2–133) yields

$$a_0 = \frac{3}{4}\frac{u'''/k}{L^2 + l^2}. \tag{2–135}$$

Combining Eqs. (2–134) and (2–135) and rearranging gives the first-order polynomial Ritz profile of the desired temperature distribution in the form

$$\frac{T(x, y) - T_\infty}{u'''l^2/k} = \frac{3}{4}\frac{[1 - (x/L)^2][1 - (y/l)^2]}{1 + (l/L)^2}. \tag{2–136}$$

Let us now comment on the accuracy of this approximate solution. Since the boundary conditions are exactly satisfied, the maximum error is anticipated at the location farthest from the boundaries, namely, at the origin of the coordinate system. And in fact, when we insert $x/L = y/l = 0$ and a specific value of l/L, say $l/L = 1$, into Eq. (2–136), we find the error to be 27.3%, an appreciable amount.† However, this error may be reduced greatly by the second-order approximation, which will be considered later. (See Example 4–11.)

We may also use an alternative procedure, the so-called *Kantorovich method*, for the selection of approximate profiles. This method is based on a generalization of the Ritz procedure. Again assume that a product solution exists composed of functions depending on only one independent variable. One of these functions, the *parameter function*,‡ is left unspecified. The new profile satisfies

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* Product forms of higher orders depend on more than one parameter.

† This error is evaluated by comparing Eq. (2–136) with the exact solution, Eq. (4–133), obtained by solving the differential formulation of the problem.

‡ In the Ritz method the term "parameter" refers to a constant parameter.

the boundary conditions of the problem only in the direction of specified functions, and when substituted into the integral formulation it yields a differential equation in terms of the parameter function. The integration constants of the solution of this differential equation are determined according to the boundary conditions in the direction of the parameter function. As we shall see in Examples 2–2 and 2–3, the Kantorovich method is especially convenient for unsteady problems.

If we return now to the specific problem under study and leave, for example, the x-direction of Eq. (2–134) unspecified,* the first-order polynomial *Kantorovich profile* becomes

$$T(x, y) - T_\infty = (l^2 - y^2)X(x), \tag{2–137}$$

which satisfies the boundary conditions only in the y-direction.

Introducing Eq. (2–137) into Eq. (2–133)† and integrating the latter with respect to y yields

$$\int_0^L (\tfrac{2}{3}l^2X'' - 2X + u'''/k)\, dx = 0. \tag{2–138}$$

Since Eq. (2–138) is true for an arbitrary length L, the integrand itself must vanish everywhere in the interval $(0, L)$. Thus the parameter function $X(x)$ satisfies the differential equation

$$X'' - (3/l^2)X = 3u'''/2kl^2, \tag{2–139}$$

subject to the boundary conditions in x, which have not been employed so far. These conditions may be determined as follows. Consider, for example, the boundary condition at $x = L$, $T(L, y) = T_\infty$. In terms of the product solution, this condition may be written in the form

$$T(L, y) - T_\infty = 0 = X(L)Y(y). \tag{2–140}$$

However, Eq. (2–140) cannot be valid for all values of $Y(y)$ unless $X(L) = 0$. Similarly, the other condition, resulting from the symmetry of temperature, is found to be $dX(0)/dx = 0$. Thus the boundary conditions in x are

$$dX(0)/dx = 0, \qquad X(L) = 0. \tag{2–141}$$

The solution of Eq. (2–139) satisfying Eq. (2–141) is

$$X(x) = \frac{u'''}{2k}\left(1 - \frac{\cosh\,(\sqrt{3}/l)x}{\cosh\,(\sqrt{3}/l)L}\right). \tag{2–142}$$

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* As will be explained in Chapter 8, this choice cannot be made arbitrarily, for the direction to be left unspecified affects the accuracy of the procedure.

† Equation (2–133) is more suitable to the Kantorovich method than Eq. (2–132).

Finally, inserting Eq. (2–142) into Eq. (2–137) and rearranging gives the first-order polynomial Kantorovich profile for the desired temperature distribution of the heater in the form

$$\frac{T(x, y) - T_\infty}{u'''l^2/k} = \frac{1}{2}\left[1 - \left(\frac{y}{l}\right)^2\right]\left(1 - \frac{\cosh(\sqrt{3}/l)x}{\cosh(\sqrt{3}/l)L}\right). \qquad (2\text{–}143)$$

For a square plate the temperature at the origin of the coordinate system, leading to the maximum error, deviates 11.5% from that of the exact solution. As expected, this result is more accurate than that of the Ritz procedure because less arbitrary restrictions are imposed on the Kantorovich profile. The accuracy of the first-order Kantorovich profile, like that of the first-order Ritz profile, may be considerably improved by the second-order profile. (See Example 4–11.)

It is worth noting that the first-order Ritz and Kantorovich profiles evaluated by the *variational form* of the same problem yield more accurate results. Thus the aforementioned error of 27.3% under the integral-Ritz procedure may be reduced to 6.05% by use of the variational-Ritz. Similarly, the 11.5% of the integral-Kantorovich becomes 2.68% by use of the variational-Kantorovich. This justifies the study of the *variational calculus* which is the concern of Chapter 8. The reason why the calculus of variations gives more accurate results compared with those of the integral method is also explained in that chapter. (See the discussion following Example 8–6.)

Example 2–2. Consider a pool reactor (Fig. 2–33) whose core is constructed from a number of vertical fuel plates of thickness $2L$. Initially the system has the uniform temperature T_∞; then assume that the constant nuclear internal energy u''' is uniformly generated in these plates. The heat transfer coefficient between the plates and the coolant is h. The temperature of the coolant re-

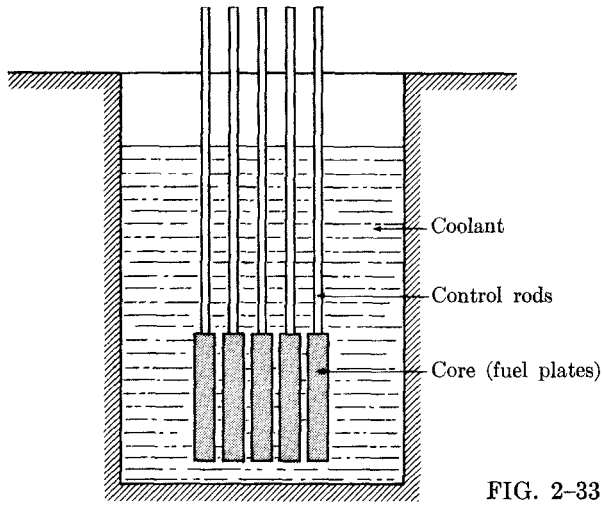


FIG. 2–33

mains constant, and the thickness of the plates is small compared with other dimensions. Thus, if the end effects are neglected, the heat transfer may be taken to be one-dimensional. We wish to formulate the unsteady temperature problem of the reactor.

The lumped, differential, and integral forms of the problem will be formulated for one of the plates of the core, again by following the five basic steps of formulation, although these steps will no longer be elaborated.

I. *Lumped formulation.* The whole plate is taken to be the system (Fig. 2–34). The lumped first law of thermodynamics, Eq. (2–16), applied to this system reduces to

$$\frac{dE}{dt} = -2Aq_n, \tag{2–144}$$

where A denotes the surface area of one side of the plate. Note that the internal energy generation u''' can no longer be identified as a power input to the system by an external electrical source. It consists, rather, of continuous changes in the composition of the nuclear fuel of which the plates are composed, as fissionable material is turned into internal energy.* These composition changes are generally small enough that thermal properties may be assumed constant.

Thus when we make use of the definition of specific heat, the left-hand side of Eq. (2–144) becomes

$$\frac{dE}{dt} = \rho(A \cdot 2L)c\frac{dT}{dt} - (A \cdot 2L)u'''. \tag{2–145}$$

Inserting Eq. (2–145) into Eq. (2–144), we get the lumped form of the first law of thermodynamics:

$$\rho cL\frac{dT}{dt} = -q_n + u'''L. \tag{2–146}$$

Since there is no need for the particular law, Eq. (2–146) is also the governing equation of the problem.

The initial and boundary conditions, respectively, are

$$T(0) = T_\infty, \tag{2–147}$$

$$q_n = h(T - T_\infty). \tag{2–148}$$

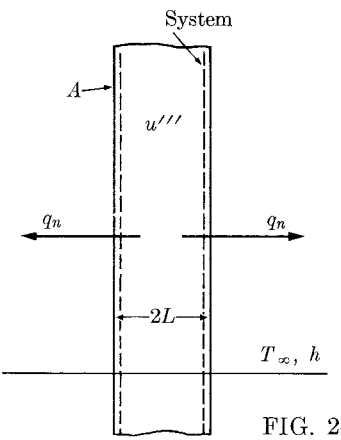


FIG. 2–34

* A similar argument pertains to the case of distributed internal energy sources and sinks resulting from exothermic and endothermic chemical reactions, respectively. See Eq. (2–12).

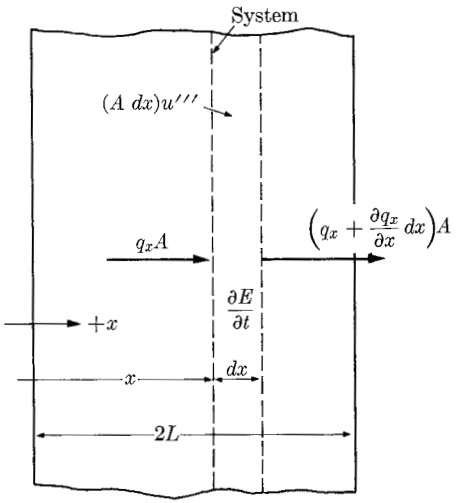


FIG. 2–35

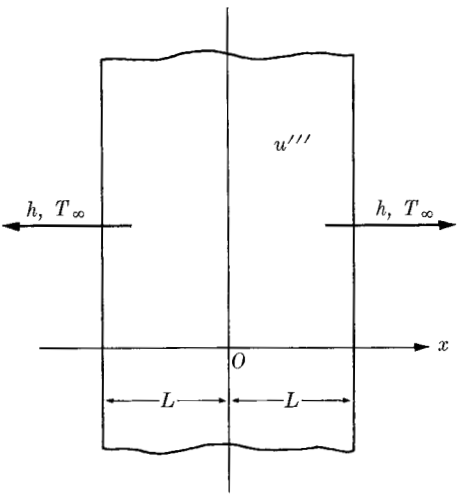


FIG. 2–36

Thus Eqs. (2–146), (2–147), and (2–148) completely describe the lumped formulation of the problem. The trivial solution of this formulation may be readily obtained by solving the combination of Eqs. (2–146) and (2–148),

$$\rho cL\frac{dT}{dt} = -h(T - T_\infty) + u'''L, \tag{2–149}$$

subject to Eq. (2–147). The result is

$$\frac{T(t) - T_\infty}{u'''L/h} = 1 - e^{-mt}, \tag{2–150}$$

where $m = h/\rho cL$.

II. *Differential formulation.* Consider the one-dimensional differential system shown in Fig. 2–35. The rightward x is assumed to be positive. The first law of thermodynamics written for Fig. 2–35 gives

$$\frac{\partial E}{\partial t} = -A\frac{\partial q_x}{\partial x}\,dx. \tag{2–151}$$

Here the time rate of change of total internal energy may be evaluated in a manner similar to that of the lumped formulation. Hence

$$\frac{\partial E}{\partial t} = \rho c(A\,dx)\frac{\partial T}{\partial t} - (A\,dx)u'''. \tag{2–152}$$

Introducing Eq. (2–152) into Eq. (2–151) and rearranging yields the appropriate form of the general law, the first law of thermodynamics, as follows:

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} + u'''. \tag{2–153}$$

Finally, considering the particular law, the x-component of Fourier's law for isotropic continua,

$$q_x = -k\frac{\partial T}{\partial x}, \tag{2–154}$$

and inserting Eq. (2–154) into Eq. (2–153), we find that the governing equation of the problem is

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + u''', \tag{2–155}$$

which for constant k reduces to

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + \frac{u'''}{\rho c}. \tag{2–156}$$

Equations (2–155) and (2–156) are the one-dimensional cartesian forms of the differential conduction equations, Eqs. (2–88) and (2–91), respectively.

The origin of the coordinate axis must now be determined, before we can state the initial and boundary conditions. The thermal and geometric symmetry of the problem suggests the middle plane of the plate to be the origin of x (Fig. 2–36). In terms of this coordinate the appropriate initial and boundary conditions are:

$$\begin{aligned} &\text{Initial:} && T(x, 0) = T_\infty, \\ &\text{Boundary:} && \frac{\partial T(0, t)}{\partial x} = 0, \qquad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty], \end{aligned} \tag{2–157}$$

Thus Eq. (2–155) or Eq. (2–156), subject to Eq. (2–157), completes the differential formulation of the problem. The solution of this formulation requires further mathematics and is left to Chapter 5. (See Example 5–4 and Problem 5–8.)

III. *Integral formulation.* Let us reconsider the systems employed for the lumped and differential formulations, and specify the origin of the coordinate axis at this step of the formulation. The first law of thermodynamics, when applied to the lumped system of Fig. 2–37, results in the previously obtained relation

$$\frac{dE}{dt} = -2Aq_n. \tag{2–144}$$

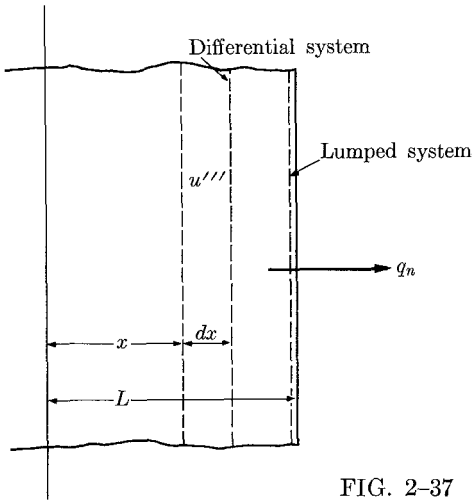


FIG. 2–37

Evaluating dE/dt in terms of the differential system integrated over the thickness of the plate, and noting the symmetry, we find that the integral form of the first law of thermodynamics is

$$\frac{d}{dt}\int_0^L \rho cT\,dx - u'''L = -q_n. \tag{2–158}$$

The particular law of the differential formulation, Eq. (2–154), being also valid for the integral formulation, gives

$$q_n = q_{x=L} = -k\left(\frac{\partial T}{\partial x}\right)_{x=L}. \tag{2–159}$$

Thus, inserting Eq. (2–159) into Eq. (2–158), we obtain the governing equation:

$$\frac{d}{dt}\int_0^L \rho cT\,dx - u'''L = k\left(\frac{\partial T}{\partial x}\right)_{x=L}. \tag{2–160}$$

The initial and boundary conditions are identical to those of the differential formulation. Hence Eq. (2–160), together with Eq. (2–157), completes the integral formulation of the problem.

An approximate solution of the foregoing formulation may now be obtained. Let us first consider the simple case $h \to \infty$. Although the spacewise temperature distribution of an unsteady problem may be easily approximated, its timewise variation is often difficult to guess. For these problems, the Kantorovich profile, being expressible in terms of an unspecified parameter function in time, becomes important. More specifically, if an unsteady problem asymptotically tends to a steady solution, and if the unsteady profile within a scale factor resembles the steady distribution,* the Kantorovich profile of the problem may be conveniently constructed from the steady solution. Let us illustrate the point in terms of the problem under consideration. Leaving the details of the trivial steady solution of the problem to the reader, we consider its result in terms of Fig. 2–36:

$$\frac{T(x) - T_\infty}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right]. \tag{2–161}$$

An unsteady first-order Kantorovich profile may now be assumed in the form

$$\frac{T(x,t) - T_\infty}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right]\tau_0(t). \tag{2–162}$$

---

* The Kantorovich profile for problems whose unsteady temperature variations do not resemble their steady distributions will be explained in Example 2–3.

Furthermore, the unsteady scale factor $\tau_0(t)$ may conveniently be taken as the unspecified parameter function to be determined. Thus introducing Eq. (2–162) into Eq. (2–160), integrating the latter with the assumption of constant properties, and rearranging gives the differential equation

$$\frac{d\tau_0}{dt} + \frac{3a}{L^2}(\tau_0 - 1) = 0, \tag{2–163}$$

subject to the condition

$$\tau_0(0) = 0. \tag{2–164}$$

The solution of Eq. (2–163) satisfying Eq. (2–164) is

$$\tau_0(t) = 1 - \exp(-3at/L^2). \tag{2–165}$$

Inserting Eq. (2–165) into Eq. (2–162), we obtain the first-order Kantorovich profile for the temperature variation of the plate in the form

$$\frac{T(x, t) - T_\infty}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right]\left[1 - \exp\left(-3\,\frac{at}{L^2}\right)\right]. \tag{2–166}$$

As $t \to \infty$ Eq. (2–166) approaches the steady solution of the problem, Eq. (2–161).

The case of finite h can be treated in the same way. Again multiplying the steady solution of the problem by the unspecified parameter function $\tau_0(t)$, we have

$$\frac{T(x, t) - T_\infty}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2 + \frac{2}{\mathrm{Bi}}\right]\tau_0(t), \tag{2–167}$$

where $\mathrm{Bi} = hL/k$. [Does $\tau_0(t)$ of Eq. (2–167) have any physical significance?] Inserting Eq. (2–167) into Eq. (2–160), and integrating the latter, we obtain the differential equation

$$\frac{d\tau_0}{dt} + \frac{3a}{L^2}\left(\frac{\mathrm{Bi}}{\mathrm{Bi} + 3}\right)(\tau_0 - 1) = 0, \tag{2–168}$$

subject to the condition

$$\tau_0(0) = 0. \tag{2–164}$$

The solution of Eq. (2–168), first satisfied by Eq. (2–164), then introduced into Eq. (2–167), gives the first-order Kantorovich profile of our problem. Thus we have

$$\frac{T(x, t) - T_\infty}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2 + \frac{2}{\mathrm{Bi}}\right]\left[1 - \exp\left(-\frac{3a}{L^2}\,\frac{\mathrm{Bi}}{\mathrm{Bi} + 3}\,t\right)\right]. \tag{2–169}$$

[What is the limiting form of Eq. (2–169) as $\mathrm{Bi} \to \infty$?]

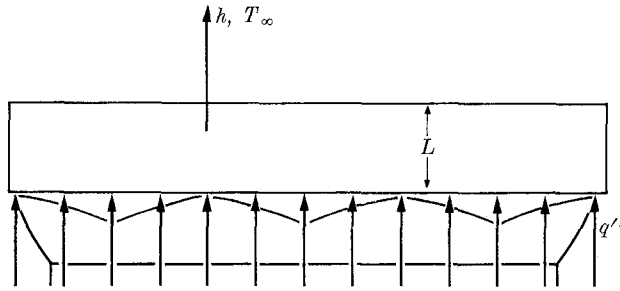


FIG. 2–38

The following alternative procedure for the selection of approximate profiles is suggested to the reader for further exercise. Write a Kantorovich profile by assuming the middle plane and surface temperatures of the plate as unspecified parameter functions to be determined. Then write the temperature of the plate corresponding to the case of finite h, in terms of a two-parameter profile. Next, utilizing the surface boundary condition, eliminate one of the parameter functions of this profile. Compare the result with Eq. (2–167).

In the next example, our primary interest lies, not in the lumped and differential formulations of the problem, but rather in the integral formulation, which requires that we define a new concept, the *penetration depth.*

Example 2–3. A hot plate of thickness L (Fig. 2–38) initially assumes the ambient temperature T_∞. From this condition, the bottom of the plate is subjected to the uniform heat flux q''. The upward heat transfer coefficient is h. The thickness L of the plate is small compared with its other dimensions, such that the heat loss from the sides may be neglected. We wish to formulate the unsteady temperature problem.

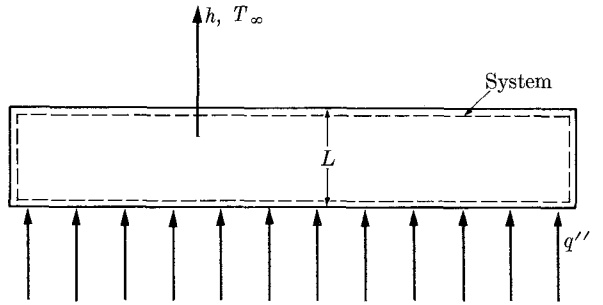


FIG. 2–39

I. *Lumped formulation.* Applying the first law of thermodynamics, combined with the definition of heat transfer coefficient, to the lumped system of Fig. 2–39 results in the lumped formulation of the problem as follows:

$$\rho cL \frac{dT}{dt} = q'' - h(T - T_\infty) \tag{2–170}$$

and

$$T(0) = T_\infty. \tag{2–171}$$

The solution of Eq. (2–170) satisfying Eq. (2–171) is

$$\frac{T(t) - T_\infty}{q''/h} = 1 - e^{-mt}, \tag{2–172}$$

where $m = h/\rho cL$.

II. *Differential formulation.* The first law of thermodynamics and Fourier's law of conduction, combined for the differential system of Fig. 2–40, result in

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right), \tag{2–173}$$

which for constant k reduces to

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}. \tag{2–174}$$

The initial and boundary conditions in x measured from the bottom* upward are

$$\begin{aligned} T(x, 0) &= T_\infty; \qquad -k \frac{\partial T(0, t)}{\partial x} = q'', \\ -k \frac{\partial T(L, t)}{\partial x} &= h[T(L, t) - T_\infty]. \end{aligned} \tag{2–175}$$

These, combined with Eq. (2–173) or Eq. (2–174), complete the differential formulation of the problem. The solution of this formulation is left to Chapter 5. [See Eq. (5–34).]

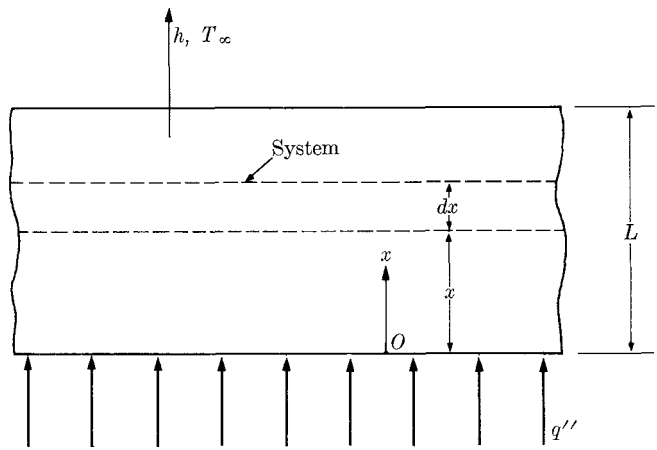


FIG. 2–40

* This is the first problem for which the appropriate coordinate axis is not obvious. The selection of the most suitable reference frame is important in view of the complexity of solution. This question will be clarified in Chapters 3 and 4.

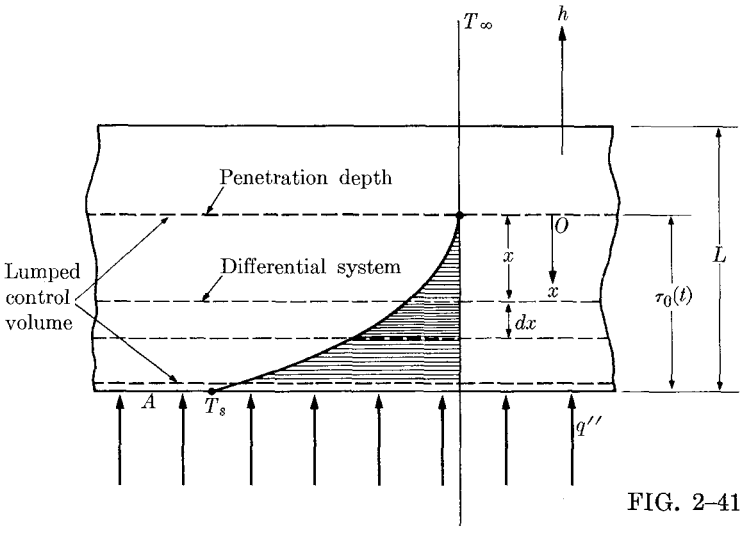


FIG. 2–41

III. *Integral formulation.* Let us first divide the problem into two domains with respect to time, such that the first time domain $0 \leq t \leq t_0$ terminates when the effect of applied heat flux reaches the upper surface; the second domain $t \geq t_0$ is valid for the remaining part of the transient while the temperature of the upper surface rises to its steady value.

The first domain of the problem can be best described by the *penetration depth* of the temperature,* $\tau_0(t)$, shown in Fig. 2–41. In the same figure the appropriate *lumped control volume* and the differential system are also indicated. The coordinate axis measured from the moving penetration depth downward proves to be convenient for the construction of the temperature profile. [See Eq. (2–181).]

The first law of thermodynamics applied to the lumped control volume of Fig. 2–41 yields

$$\frac{dE}{dt} = \rho c A T_\infty \frac{d\tau_0}{dt} + q'' A, \tag{2–176}$$

where A is the surface area of one side of the plate. Expressing the total internal energy E of the lumped control volume in terms of the internal energy of the differential system gives the appropriate integral formulation of the general law:

$$\frac{d}{dt}\int_0^{\tau_0(t)} \rho c T \, dx = \rho c T_\infty \frac{d\tau_0}{dt} + q''. \tag{2–177}$$

Although the problem is distributed, none of the terms of Eq. (2–177) requires

* This is analogous to the concept of velocity and temperature boundary-layer thicknesses of the boundary-layer theory.

the use of any particular law. However, the x-component of Fourier's particular law will be used later in connection with boundary conditions. [See Eq. (2–179).] Thus the governing equation of the first domain is identical to Eq. (2–177).

The uniform initial temperature of the plate gives the initial condition

$$T(x, 0) = T_\infty. \tag{2–178}$$

The boundary condition on the lower surface of the plate, relating the constant heat flux q'' to the temperature by the particular law, may be written in the form

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=\tau_0(t)} = -q''. \tag{2–179}$$

Since in this time domain the penetration depth is less than the thickness of the plate, the boundary condition expressing the heat transfer from the upper surface is not valid. Instead, zero heat flux across the plane surface of the penetration depth

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = 0 \tag{2–180}$$

should be used. Thus Eqs. (2–177), (2–178), (2–179), and (2–180) complete the integral formulation of the problem in the first time domain. Before proceeding with the formulation of the second time domain, however, let us find an approximate solution for the first time domain.

Since the unsteady temperature variation in this domain approaches, not a known steady temperature, but the unknown initial temperature of the second domain, the unsteady Kantorovich profile cannot be constructed from the steady solution. (Note the Kantorovich profile of Example 2–2.) This suggests a second approach in constructing the unsteady Kantorovich profiles, one which requires the selection of functions (polynomial, circular, etc.) satisfying the boundary conditions. Although the order of the polynomial (or the type of circular function) is to a large extent arbitrary, simplicity may be used as a guide for the construction of these profiles. Thus, for example, the polynomial of least order satisfying the boundary conditions is generally the best approximation of the actual profile. In the present case we may assume the parabola

$$T(x, t) - T_\infty = \left(\frac{q''}{2k}\right)\frac{x^2}{\tau_0(t)}, \tag{2–181}$$

which satisfies the boundary conditions given by Eqs. (2–179) and (2–180). Introducing Eq. (2–181) into Eq. (2–177), and integrating the latter with the assumption of constant properties, we obtain the trivial but *nonlinear* differential equation

$$d\tau_0^2 = 6a\, dt, \tag{2–182}$$

subject to the initial condition

$$\tau_0(0) = 0. \tag{2–183}$$

The solution of Eq. (2–182) which satisfies Eq. (2–183) is

$$\tau_0(t) = (6at)^{1/2}. \tag{2–184}$$

Thus the first-order Kantorovich profile giving the temperature variation in the first time domain of the problem is found to be

$$T(x, t) - T_\infty = \left(\frac{q''}{2k}\right) \frac{x^2}{(6at)^{1/2}}, \qquad \text{when} \qquad 0 \leq t \leq t_0. \tag{2–185}$$

Furthermore, inserting $\tau_0(t_0) = L$ into Eq. (2–184), we get the penetration time of the applied heat flux q'' to the upper surface of the plate as

$$t_0 = L^2/6a. \tag{2–186}$$

For a comparison among the following three different solutions of the problem, let us consider, for example, the temperature variation at the bottom of the plate. From the solution under study, introducing $x = \tau_0(t) = (6at)^{1/2}$ into Eq. (2–185), we obtain

$$T_s(t) - T_\infty = \left(\frac{3}{2}\right)^{1/2} \left(\frac{q''}{k}\right) (at)^{1/2}. \tag{2–187}$$

The exact solution of the problem, which will be seen in Chapter 7*, gives

$$T_s(t) - T_\infty = \left(\frac{4}{\pi}\right)^{1/2} \left(\frac{q''}{k}\right) (at)^{1/2}. \tag{2–188}$$

Hence the error involved in Eq. (2–187) is 8.4%. If the approximate profile given by Eq. (2–181) were introduced into the variational form of the formulation (see Example 8–8), the result would be

$$T_s(t) - T_\infty = \left(\frac{5}{4}\right)^{1/2} \left(\frac{q''}{k}\right) (at)^{1/2}. \tag{2–189}$$

Equation (2–189) leads to an error of approximately 0.89%. The foregoing comparison thus indicates once more the importance of the variational calculus as an approximate method.

Having finished the study of the first time domain, we now return to the formulation of the problem for the second time domain. Consider the lumped and differential systems shown in Fig. 2–42. The coordinate axis is measured

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* The special case of Problem 7–1 corresponding to $h = 0$.

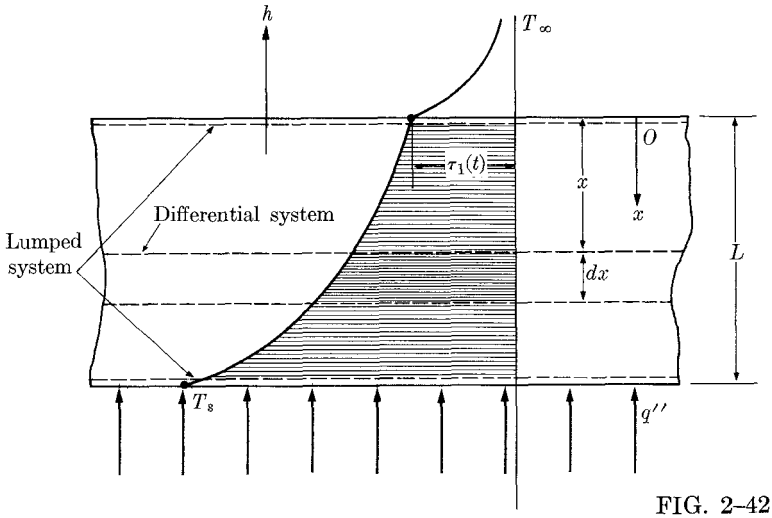


FIG. 2–42

from the upper surface downward. The first law of thermodynamics and Fourier's law of conduction result in the appropriate integral form of the governing equation:

$$\frac{d}{dt}\int_0^L \rho c T\, dx = q'' - k\left(\frac{\partial T}{\partial x}\right)_{x=0}. \tag{2–190}$$

The initial condition of this domain is the final condition of the first domain. Also, the boundary condition on the lower surface should be modified according to the new coordinate axis as

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=L} = -q''. \tag{2–191}$$

Furthermore, the boundary at the upper surface, now transferring heat to the ambient, satisfies the condition

$$+k\left(\frac{\partial T}{\partial x}\right)_{x=0} = h(T_{x=0} - T_\infty). \tag{2–192}$$

Thus Eqs. (2–190), (2–185), for $t = t_0$, (2–191), and (2–192) describe the integral formulation of the problem for the second time domain.

Finally, let us find an approximate temperature variation for the second time domain. Although this problem, like that of Example 2–2, approaches a steady solution at the end of the second time domain, an unsteady Kantorovich profile cannot be constructed by scaling the exact steady solution of the problem in terms of an unspecified parameter function. In this case a procedure analogous to that of the first time domain is followed. The simplest possible profile is con-

structed from the combination of an unspecified parameter function in time and appropriately selected space functions satisfying the boundary conditions of the problem. It is important to note that this profile must approach that of the first time domain as $t \to t_0$, and of the steady solution as $t \to \infty$.

Let us assume the parabolic profile

$$T(x, t) - T_\infty = Ax^2 + Bx + C.$$

If we take, for example, the upper surface temperature $\tau_1(t)$ as the unspecified parameter function to be determined, this profile may be written in the form

$$T(x, t) - T_\infty = \left(\frac{q''}{h} - \tau_1\right)\frac{hx^2}{2kL} + \left(1 + \frac{hx}{k}\right)\tau_1. \tag{2–193}$$

As $t \to \infty$ Eq. (2–193) approaches the steady solution of the problem,

$$T(x) - T_\infty = \left(1 + \frac{hx}{k}\right)\frac{q''}{h},$$

and it becomes identical to the solution of the first time domain as $t \to t_0$.

Inserting Eq. (2–193) into Eq. (2–190) and integrating the latter with the assumption of constant properties, we obtain the *linear* differential equation

$$\frac{d\tau_1}{dt} + \left(\frac{m}{1 + \text{Bi}/3}\right)\left(\tau_1 - \frac{n}{m}\right) = 0, \tag{2–194}$$

subject to the initial condition

$$\tau_1(t_0) = 0. \tag{2–195}$$

Here $m = h/\rho cL$, $n = q''/\rho cL$, and $\text{Bi} = hL/k$.

The solution of Eq. (2–194) satisfying Eq. (2–195) is

$$\tau_1(t) = \frac{n}{m}\left\{1 - \exp\left[-\frac{m(t - t_0)}{1 + \text{Bi}/3}\right]\right\}. \tag{2–196}$$

Finally, combining Eqs. (2–196) and (2–193) and rearranging gives the first-order Kantorovich profile for the second time domain of the problem in the form

$$\frac{T(x, t) - T_\infty}{q''L/k} = \frac{1}{2}\left(\frac{x}{L}\right)^2 + \left[\frac{1}{\text{Bi}} + \left(1 - \frac{x}{2L}\right)\frac{x}{L}\right]\left\{1 - \exp\left[-\frac{m(t - t_0)}{1 + \text{Bi}/3}\right]\right\}, \tag{2–197}$$

when $t \geq t_0$.

So far we have considered three simple problems that were so stated as to include the complete information for the formulation. Our main objective has been to develop the ability to formulate problems in the lumped, differential, and integral forms. In a given physical situation, however, the formulation of a

problem often requires that we make a number of assumptions as part of the formulation. For this reason, in the next two examples only the physics of the problem is described. The necessary information for formulation is then given in the course of formulation.

Example 2–4. Two rigid circular disks are coaxially pressed together by the external load P as shown in Fig. 2–43. Initially the system is stationary and at the ambient temperature T_∞. Then the upper plate suddenly assumes the constant angular velocity ω. The coefficient of dry friction between the disks is μ. How should we formulate the problem?

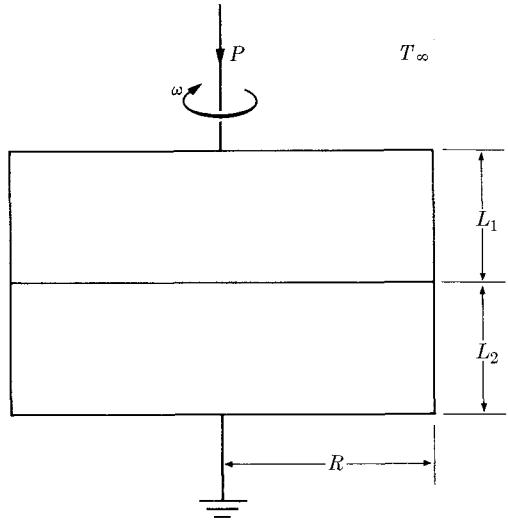


FIG. 2–43

Note first that the coefficients of upward, downward, and horizontal heat transfer resulting from gravitational free convection are different. Second, the heat transfer from the upper disk is increased many times by centrifugal free convection. Thus we have four different heat transfer coefficients, the upward h_1 and horizontal h_3 for the rotating disk, the downward h_2 and horizontal h_4 for the stationary disk. To simplify the formulation, the disks are assumed to be homogeneous and isotropic.

I. *Differential formulation.* The properties of the upper and lower disks are distinguished by the subscripts 1 and 2. The first law of thermodynamics and Fourier's law of conduction written for the two-dimensional cylindrical systems of Fig. 2–44 give the differential formulation of the problem as follows:

$$\frac{\partial T_1}{\partial t} = a_1\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) + \frac{\partial^2 T_1}{\partial z^2}\right], \tag{2–198}$$

$$\frac{\partial T_2}{\partial t} = a_2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) + \frac{\partial^2 T_2}{\partial z^2}\right], \tag{2–199}$$

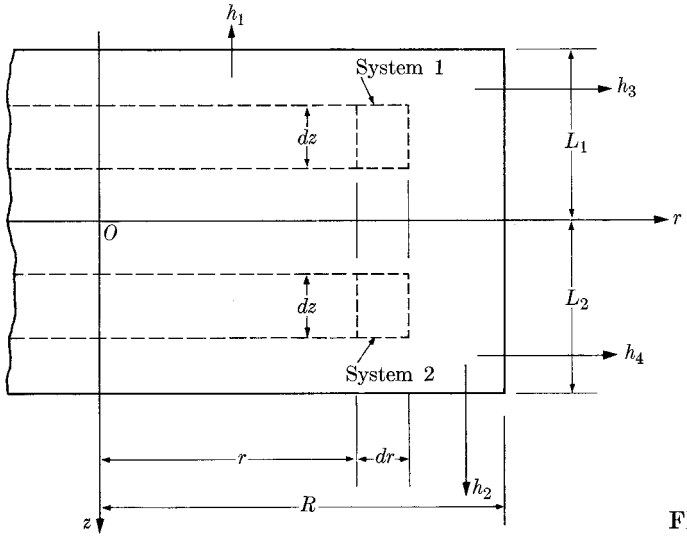


FIG. 2–44

with two initial and eight boundary conditions:*

$$T_1(r, z, 0) = T_\infty, \tag{2–200}$$

$$T_2(r, z, 0) = T_\infty, \tag{2–201}$$

$$+k_1 \frac{\partial T_1(r, -L_1, t)}{\partial z} = h_1[T_1(r, -L_1, t) - T_\infty], \tag{2–202}$$

$$T_1(r, 0, t) = T_2(r, 0, t), \tag{2–203}$$

$$-k_1 \frac{\partial T_1(r, 0, t)}{\partial z} + \mu p(r)\omega r = -k_2 \frac{\partial T_2(r, 0, t)}{\partial z}, \tag{2–204}$$

$$-k_2 \frac{\partial T_2(r, L_2, t)}{\partial z} = h_2[T_2(r, L_2, t) - T_\infty], \tag{2–205}$$

$$\frac{\partial T_1(0, z, t)}{\partial r} = 0, \tag{2–206}$$

$$-k_1 \frac{\partial T_1(R, z, t)}{\partial r} = h_3[T_1(R, z, t) - T_\infty], \tag{2–207}$$

$$\frac{\partial T_2(0, z, t)}{\partial r} = 0, \tag{2–208}$$

$$-k_2 \frac{\partial T_2(R, z, t)}{\partial r} = h_4[T_2(R, z, t) - T_\infty], \tag{2–209}$$

where $p(r)$ is the local pressure between the disks.

* As with the second time domain of Example 2–3, the selection of axial origin is irrelevant for the present formulation and will not be discussed here.

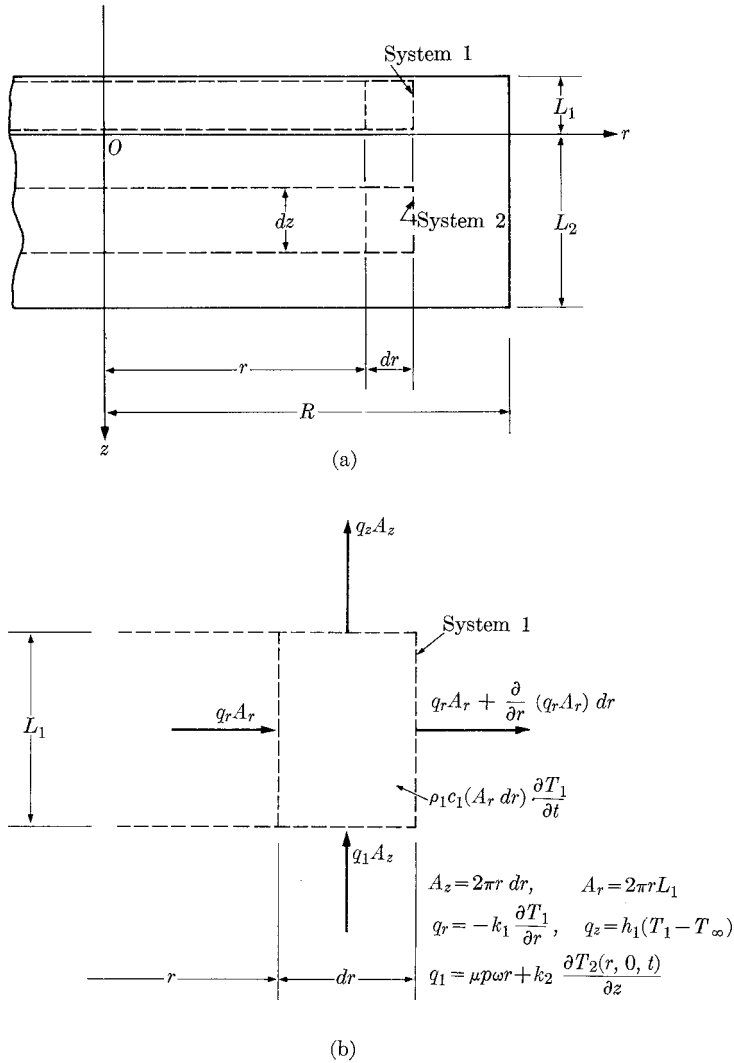


FIG. 2–45

The foregoing differential formulation, being classical, is easy to establish, but it is difficult or even impossible to solve. On the other hand, whenever physics permits there may be simpler formulations of the same problem, possibly leading to a solution. Let us demonstrate now how certain simplifications may be brought into the differential formulation.

(a) *Axially lumped upper disk.* When $k_1 \gg k_2$ or $L_1 \ll L_2$, the axial temperature variation of the upper disk may be lumped. Thus the first law of thermodynamics, Fourier's law of conduction, and the definition of heat transfer coefficient applied to system 1, Fig. 2–45, yield the equation of conduction

for the upper disk as follows:

$$\frac{\partial T_1}{\partial t} = \frac{a_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) - \frac{h_1}{\rho_1 c_1 L_1}(T_1 - T_\infty) + \frac{\mu p(r)\omega r}{\rho_1 c_1 L_1} - \left[-k_2\frac{\partial T_2(r, 0, t)}{\partial z}\right]. \tag{2–210}$$

However, noting the equality of interface temperatures,

$$T_1(r, t) = T_2(r, 0, t), \tag{2–211}$$

we may transform Eq. (2–210) to a boundary condition for the lower disk. [See Eq. (2–212.)] The equation of conduction for the lower disk remains unchanged. Thus the formulation of the problem for the present case becomes

$$\frac{\partial T_2}{\partial t} = a_2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) + \frac{\partial^2 T_2}{\partial z^2}\right], \tag{2–199}$$

subject to

$$T_2(r, z, 0) = T_\infty, \tag{2–201}$$

$$\frac{\partial T_2(r, 0, t)}{\partial t} = \frac{a_1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_2(r, 0, t)}{\partial r}\right] - \frac{h_1}{\rho_1 c_1 L_1}[T_2(r, 0, t) - T_\infty] + \frac{\mu p\omega r}{\rho_1 c_1 L_1} - \frac{1}{\rho_1 c_1 L_1}\left[-k_2\frac{\partial T_2(r, 0, t)}{\partial z}\right], \tag{2–212}$$

$$-k_2\frac{\partial T_2(r, L_2, t)}{\partial z} = h_2[T_2(r, L_2, t) - T_\infty], \tag{2–205}$$

$$\frac{\partial T_2(0, z, t)}{\partial r} = 0, \tag{2–208}$$

$$-k_4\frac{\partial T_2(R, z, t)}{\partial r} = h_4[T_2(R, z, t) - T_\infty]. \tag{2–209}$$

Note that the heat transfer coefficient h_3 has disappeared from the formulation of the problem. (What boundary condition is then satisfied at the peripheral surface of the upper disk? Is it possible to assume insulation on this surface?)

For the limiting case $L_1 \to 0$, the present formulation remains unchanged except for the boundary condition given by Eq. (2–212), which simplifies to

$$0 = -h_1[T_2(r, 0, t) - T_\infty] + \mu p\omega r - \left[-k_2\frac{\partial T_2(r, 0, t)}{\partial z}\right]. \tag{2–213}$$

We thus learn that by neglecting axial temperature variation in the upper disk only, the formulation of the problem is reduced from two partial differential equations and two initial plus eight boundary conditions to one partial differential equation and one initial plus four boundary conditions.

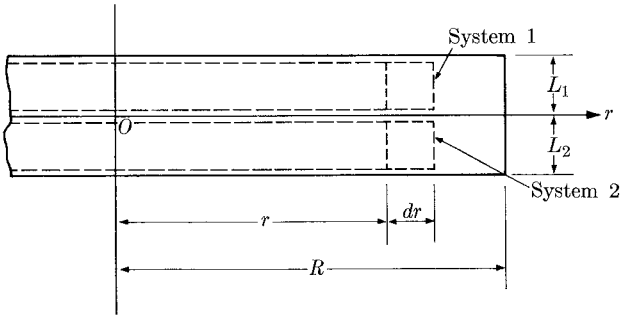


FIG. 2–46

(b) *Both disks axially lumped.* When both disks are thin such that L_1 and $L_2 \ll R$, or when k_1 and k_2 are large, the axial temperature variation in both disks may be neglected.

Thus, considering the axially lumped and radially differential systems shown in Fig. 2–46, we may obtain a single equation of conduction in the form

$$\frac{\partial T}{\partial t} = \frac{\bar{a}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{\bar{h}}{\overline{\rho cL}}(T - T_\infty) + \frac{\mu p \omega r}{2\overline{\rho cL}}, \tag{2–214}$$

where T is the common temperature of the disks, $\bar{a} = \overline{kL}/\overline{\rho cL}$, $\overline{kL} = (k_1L_1 + k_2L_2)/2$, $\overline{\rho cL} = (\rho_1c_1L_1 + \rho_2c_2L_2)/2$, and $\bar{h} = (h_1 + h_2)/2$.

The initial and boundary conditions to be imposed on Eq. (2–214) are

$$T(r, 0) = T_\infty, \tag{2–215}$$

$$\frac{\partial T(0, t)}{\partial r} = 0, \tag{2–216}$$

$$-\bar{k}\frac{\partial T(R, t)}{\partial r} = \bar{h}[T(R, t) - T_\infty], \tag{2–217}$$

where $\bar{k} = (k_1 + k_2)/2$. It is, of course, possible to replace Eq. (2–217) with the approximate condition

$$\frac{\partial T(R, t)}{\partial r} \cong 0. \tag{2–218}$$

II. *Integral formulation.* A two-dimensional integral formulation of the problem, involving a penetration surface which is no longer plane, is beyond the scope of the present discussion. The one-dimensional integral formulation corresponding to case (b), however, is trivial and is left to the reader as an exercise.

The solutions of the foregoing four formulations, which do not concern us here, require that the pressure distribution $p(r)$ between the disks be specified. This determines the work done by friction. Two commonly assumed cases are

(i) constant pressure, $p = \text{const}$,

(ii) constant wear, $pr = \text{const}$.

The latter of these is physically more realistic, and also turns out to be more convenient for obtaining a solution.*

Example 2–5. An insulated thin-walled vessel initially contains superheated water vapor at temperature T_v (Fig. 2–47). Then the external surface of the bottom of the vessel is exposed to the surrounding air at temperature $T_\infty (T_\infty < T_v)$. We wish to formulate the problem in terms of the instantaneous thickness $X(t)$ of the condensate and the other variables involved.

The following assumptions and facts may be used in the formulation of the problem:

(a) The condensate-vapor interface is at the saturation temperature T_s.

(b) The problem is one-dimensional because of the peripheral insulation of the vessel.

(c) The temperature drop across the bottom thickness of the vessel may be neglected.

(d) The properties of the vapor and condensate are constant, and are distinguished by the subscripts 1 and 2, respectively.

(e) The constant density difference between the vapor and condensate causes a downward flow of vapor at the uniform velocity†

$$V = \left(\frac{\rho_2}{\rho_1} - 1\right)\frac{dX}{dt}.$$

Hence the two-domain one-dimensional differential formulation of the problem in x, measured from the bottom of the vessel upward, may be summarized as follows:

Vapor:

$$\frac{\partial T_1}{\partial t} - \left(\frac{\rho_2}{\rho_1} - 1\right)\frac{dX}{dt}\frac{\partial T_1}{\partial x} = a_1 \frac{\partial^2 T_1}{\partial x^2},$$

$$T_1(x, 0) = T_v,$$

$$T_1(\infty, t) = T_v,$$

Condensate:

$$\frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x^2},$$

$$T_2(x, 0) = T_s,$$

$$+k_2 \frac{\partial T_2(0, t)}{\partial x} = h[T_2(0, t) - T_\infty],$$

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* Compare the two solutions of Problem 3–30 corresponding to the cases of constant pressure and wear.

† See Eq. (2–114).

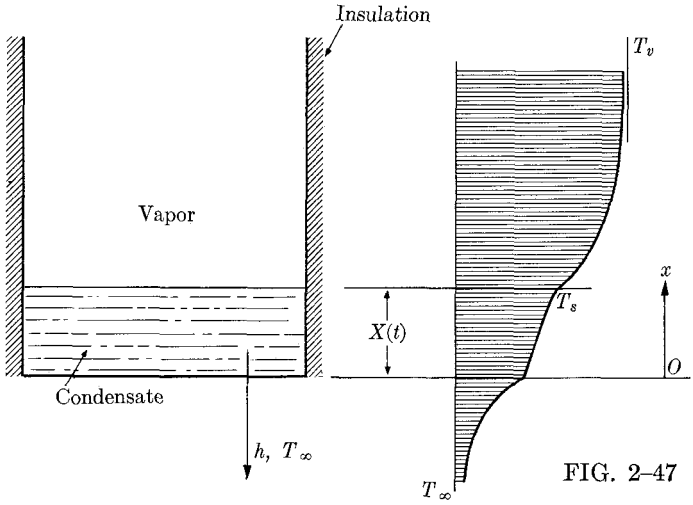


FIG. 2–47

coupled along the condensate-vapor interface by the boundary conditions

$$T_1(X, t) = T_2(X, t) = T_s,$$

$$-k_1 \frac{\partial T_1(X, t)}{\partial x} = -k_2 \frac{\partial T_2(X, t)}{\partial x} + \rho_2 h_{vc} \frac{dX}{dt}.$$

(How many boundary conditions are needed for the foregoing formulation and how many have been stated?)

The differential formulations of two-domain problems are often difficult to solve, especially when the domains are finite. With appropriate physical reasoning, however, these may be occasionally reduced to simpler problems. Let us first consider that the vapor is at the saturation temperature, or assume that the superheat is small enough that it can be neglected. Thus $T_1(x, t) = T_s$, and the formulation is reduced to that of the condensate, for which (leaving out the subscript) we have

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2},$$

$$T(x, 0) = T_s,$$

$$+k \frac{\partial T(0, t)}{\partial x} = h[T(0, t) - T_\infty], \tag{2–219}$$

$$T(X, t) = T_s,$$

$$-k \frac{\partial T(X, t)}{\partial x} + \rho h_{vc} \frac{dX}{dt} = 0.$$

Even the solution of this simplified formulation involves mathematical difficulties.

Let us next consider a quasi-steady formulation as a further simplification. Given a small rate of condensation, which is often the case, the time rate of change of internal energy of the condensate may be neglected. Thus the previous formulation is reduced to

$$0 = \frac{\partial^2 T}{\partial x^2}, \tag{2–220}$$

$$+k\frac{\partial T(0,t)}{\partial x} = h[T(0,t) - T_\infty], \tag{2–221}$$

$$T(X,t) = T_s, \tag{2–222}$$

$$-k\frac{\partial T(X,t)}{\partial x} + \rho h_{vc}\frac{dX}{dt} = 0. \tag{2–223}$$

Note that the unsteadiness of this formulation is associated only with the boundary condition given by Eq. (2–223).

The solution of the foregoing case may be readily obtained. Solving for T from Eqs. (2–220), (2–221), and (2–222), then inserting the result into Eq. (2–223) yields

$$\rho h_{vc}\frac{dX}{dt} = \frac{T_s - T_\infty}{X/k + 1/h}, \tag{2–224}$$

or

$$\frac{1}{k}X\,dX + \frac{1}{h}dX = \left(\frac{T_s - T_\infty}{\rho h_{vc}}\right)dt, \tag{2–225}$$

subject to

$$X(0) = 0. \tag{2–226}$$

The solution of Eq. (2–225) which satisfies Eq. (2–226) is

$$\frac{X(t)}{k/h} = -1 + \left[1 + \frac{2h^2(T_s - T_\infty)t}{k\rho h_{vc}}\right]^{1/2}. \tag{2–227}$$

Our final simplification is to eliminate the temperature drop across the condensate. This is a valid assumption when k is large or t is small. Hence, leaving out the thermal resistance X/k of the condensate in Eq. (2–224), we have

$$\rho h_{vc}\frac{dX}{dt} = \frac{(T_s - T_\infty)}{1/h}, \tag{2–228}$$

which satisfies Eq. (2–226) and integrates to

$$X(t) = \frac{h(T_s - T_\infty)}{\rho h_{vc}}t. \tag{2–229}$$

[What is the limiting form of Eq. (2–227) as $k \to \infty$ or $t \to 0$?]

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Problems

2–1. A chamber contains liquid and vapor water in equilibrium at a temperature a little less than the critical temperature. One pound of liquid is withdrawn through a valve at the bottom of the chamber while the chamber is surrounded by a constant-temperature bath which prevents the temperature of the contents from changing from its initial value (Fig. 2–48). The specific volume of the liquid is not negligible as compared with that of the vapor. (a) Find an expression for the increase in volume of

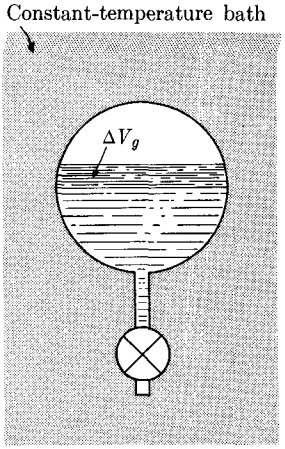


FIG. 2–48

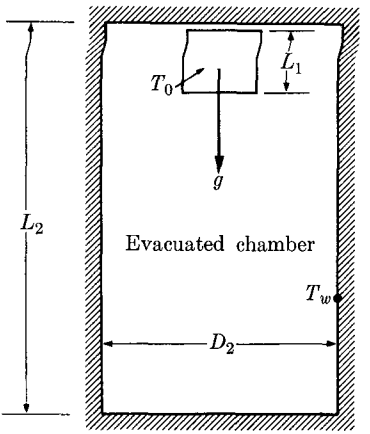


FIG. 2–49

the vapor phase in the chamber, in terms only of saturation properties of liquid and vapor water. (b) Find an expression for the heat transferred from the bath to the contents of the chamber as a result of the withdrawal of the liquid, in terms only of saturation properties.

2–2. A square copper plate of thickness δ having the initial temperature T_0 is dropped into an evacuated vertical chamber whose walls are maintained at the constant temperature $T_w(\gg T_0)$ (Fig. 2–49). Using the data given below, compute the temperature of the plate when it reaches the bottom of the chamber.

copper: $\rho_1 = 500$ lbm/ft^3; $c_1 = 0.1$ Btu/lbm · °F

$L_1 = 6$ in. $\quad \delta = \frac{1}{4}$ in.

$L_2 = 40$ ft $\quad D_2 = 10$ ft

$\epsilon_1 = 0.8 \quad T_w = 2000$°R

$\epsilon_2 = 0.4 \quad T_0 = 60$°F

$g = 32.2$ ft^2/sec

2–3. A solid rod moving through a tube melts as a result of the uniform heat flux q'' applied peripherally to the tube, and assumes a parabolic velocity profile (Fig. 2–50). The density ρ of the solid is approximately equal to the density of the fluid. The velocity of the solid is V, the temperature of the solid at the inlet is T_0, the latent heat of melting is h_{fs}, and the specific heat of the solid and fluid are c_s, c_f, respectively. The friction between the solid rod and the tube may be neglected. The axial conduction is negligible compared to the enthalpy flow in both the solid and the fluid. Assuming a parabolic radial temperature distribution for the fluid, find this distribution at the distance L from the inlet of the tube.

2–4. Consider the steady one-dimensional flows of a frictionless incompressible fluid through a constant-area tube and a diffuser of the same length (Fig. 2–51). The tube and diffuser are subjected peripherally to the same uniform heat flux q''. The

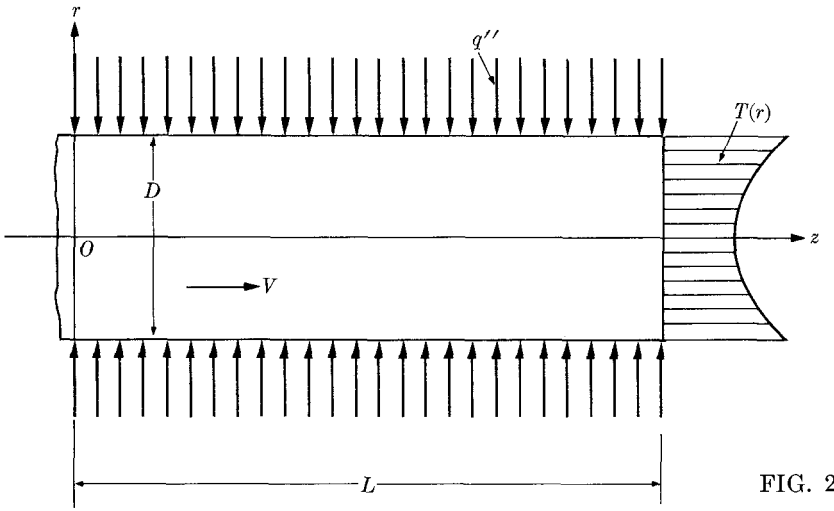


FIG. 2–50

inlet diameter and inlet velocity of the diffuser are identical to those of the tube. (a) Is the exit temperature of the diffuser higher or lower than that of the tube? Base your statement on physical reasoning rather than mathematics. (b) Support your argument with a simple analysis in which the axial conduction may be neglected.

2–5. Reconsider Example 2–5. Assume that the vapor is at the saturation temperature and that the temperature of the condensate may be lumped. Apply the first law of thermodynamics directly to the condensate. Express the heat loss from the bottom of the condensate in terms of a heat transfer coefficient. Compare the result with Eq. (2–228).

2–6. Consider a three-dimensional, differential control volume in cartesian, cylindrical, and spherical coordinates. (a) Obtain the special forms of ∇T and $\nabla^2 T$ in these coordinates. (Here T denotes any scalar property.) (b) Derive the equation of conduction for a homogeneous, isotropic, frictionless incompressible fluid by following the five basic steps of formulation. (c) Utilizing (a), write the cartesian, cylindrical, and spherical forms of Eq. (2–91). (d) Compare the results of (b) and (c).

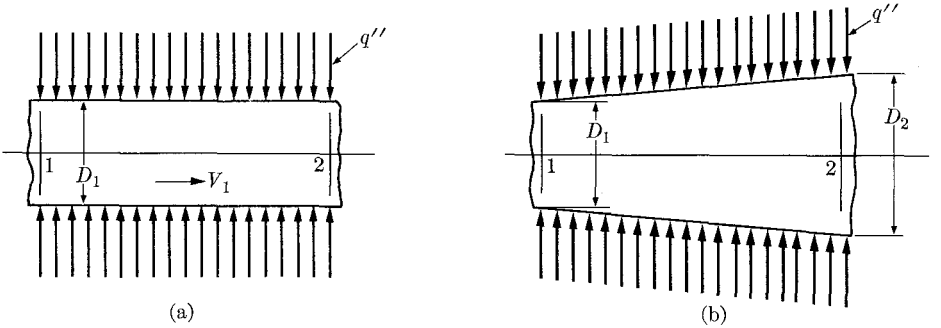


FIG. 2–51

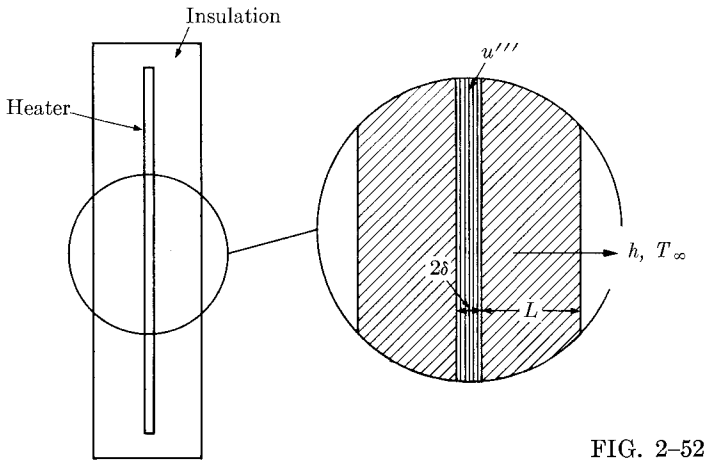


FIG. 2–52

2–7. Reconsider Example 2–1. Using the integral form of the equation of conduction and assuming (a) circular (trigonometric) functions in both directions, (b) a circular function for the shorter side but an unspecified parameter function for the longer side, find the steady temperature of the heater.

2–8. Derive the lumped and integral formulations of Example 2–3 starting from its differential formulation.

Hint: For the integral formulation suitable to the first time domain, integrate the differential formulation over the interval $(0, \tau_0)$; rearrange the result by *Leibnitz's integral formula,*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x, \xi)\, d\xi = \int_{a(x)}^{b(x)} \frac{\partial F(x, \xi)}{\partial x}\, d\xi + F(x, b)\frac{db}{dx} - F(x, a)\frac{da}{dx}.$$

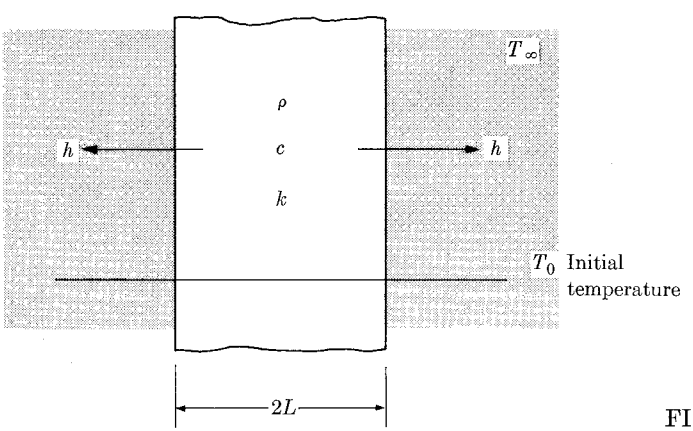


FIG. 2–53

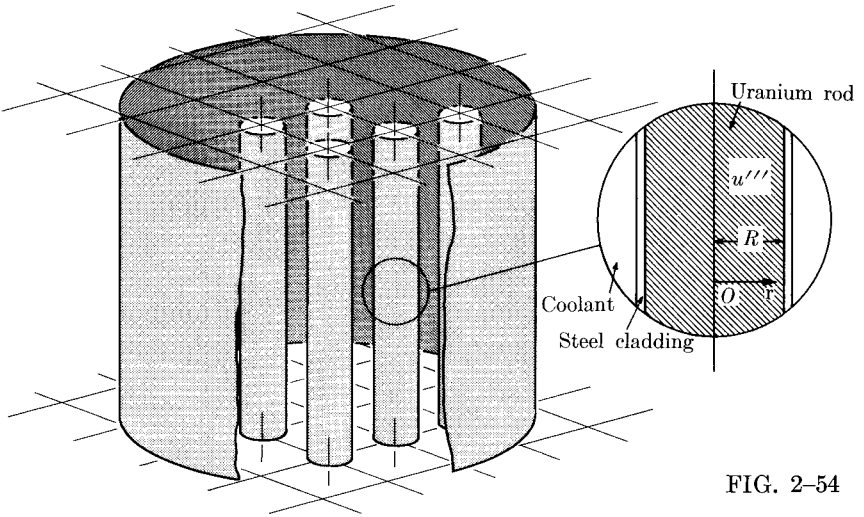


FIG. 2–54

2–9. Internal energy is to be generated electrically in a flat metal plate of thickness 2δ for heating purposes. To obtain low surface temperatures, and for electrical insulation, this plate is covered by an electrical insulator of thickness L which is also a poor thermal conductor (Fig. 2–52). The heater is at the ambient temperature T_∞ initially; then the internal energy u''' is suddenly generated in the heater. Make a one-dimensional analysis.

2–10. A flat plate of thickness $2L$ having the uniform initial temperature T_0 is plunged suddenly into a bath at constant temperature T_∞ (Fig. 2–53). The heat transfer coefficient is h. The density, heat capacity, and thermal conductivity of the plate are ρ, c and k, respectively. Formulate the problem.

2–11. The core of a pool reactor is made of cylindrical fuel elements, each composed of a uranium rod of radius R and a stainless steel cladding of negligible thickness (Fig. 2–54). The reactor has the uniform temperature T_∞ initially; then internal energy is assumed suddenly to be generated in the uranium rods as

$$u'''/u_0''' = 1 - (r/R)^2,$$

where u_0''' is the internal energy generation at the center line. The temperature of the coolant is held constant at T_∞. The heat transfer coefficient is large. Formulate the problem.

2–12. A solid sphere of radius R having the initial temperature T_0 is dropped into boiling water at temperature T_∞. Formulate the problem.

2–13. Reformulate the unsteady problem associated with Example 2–1 for a large heat transfer coefficient. Assume that the heater is at the ambient temperature T_∞ initially and that the uniform internal energy u''' is suddenly generated in the heater.

2–14. Consider the lumped formulation of Examples 2–2 and 2–3. Evaluate the total entropy generation in these examples.

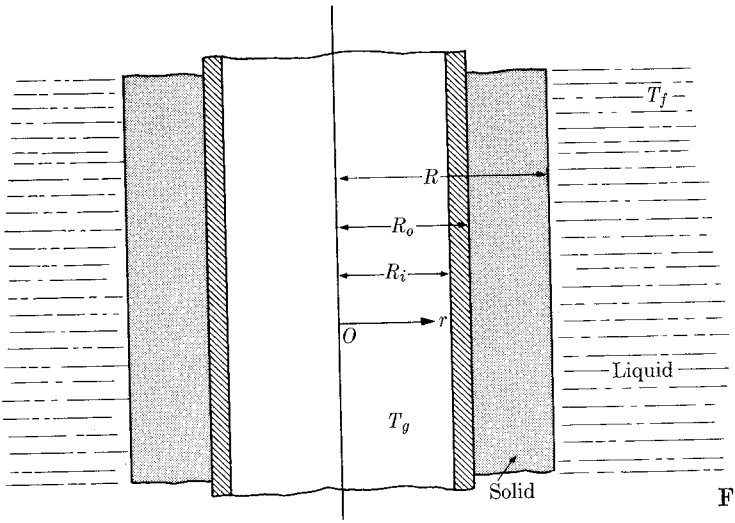


FIG. 2–55

2–15. A liquid at the freezing temperature T_f is being frozen around a tube cooled by an internal flow of gas at temperature T_g (Fig. 2–55). Formulate the problem in terms of the instantaneous location R of the cylindrical solid-liquid interface and the other parameters involved.*

2–16. Reconsider the differential formulation of Example 2–5 given by Eq. (2–219). (a) Find the corresponding integral formulation by following the physical approach. (b) Obtain an approximate solution of the problem in terms of polynomials.

2–17. Consider the unsteady one-dimensional flow of a frictionless incompressible fluid through a diffuser. The diffuser is subjected to the uniform peripheral heat flux q'' (Fig. 2–56). Find the local and total entropy generation in this flow, assuming that the axial conduction is (a) negligible, and (b) not negligible.

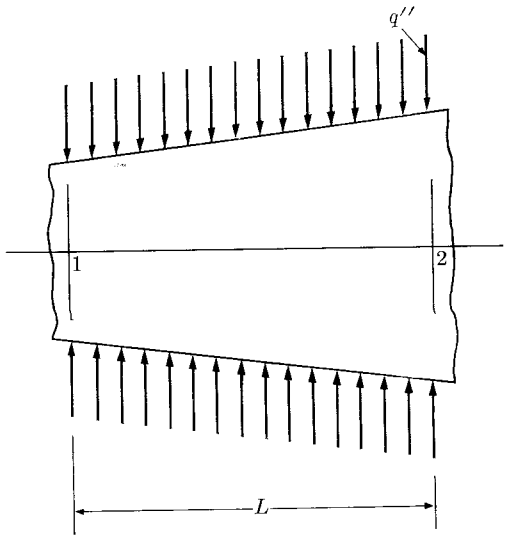


FIG. 2–56

2–18. Consider a two-dimensional homogeneous anisotropic body, say a flat plate of laminated material as shown in Fig. 2–57. The laminations of the plate make an angle α with the normal n to the surfaces. Assume that the maximum and minimum values of the thermal conductivity, say k_ξ and k_η, are in the directions of ξ and η, respectively. (These directions and the corresponding values of the conductivity are called the *principal axes* and *the principal values*, respectively. It may be shown that the values of the conductivity in other directions vary as an

* See Reference 17.

ellipse whose axes are the principal values of the conductivity.) The components of the heat flux vector in ξ and η are

$$q_\xi = -k_\xi \frac{\partial T}{\partial \xi} \quad \text{and} \quad q_\eta = -k_\eta \frac{\partial T}{\partial \eta}.$$

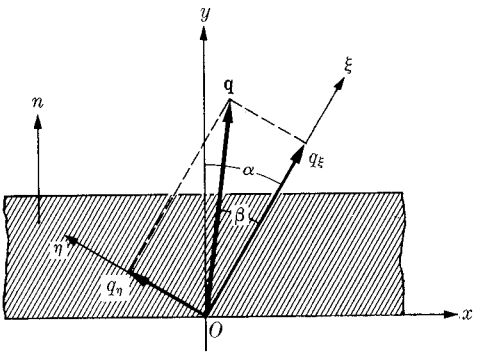


FIG. 2–57

(a) Show that the components of the heat flux vector in x and y are

$$q_x = -k_\xi \sin\alpha \frac{\partial T}{\partial \xi} + k_\eta \cos\alpha \frac{\partial T}{\partial \eta},$$

$$q_y = -k_\xi \cos\alpha \frac{\partial T}{\partial \xi} - k_\eta \sin\alpha \frac{\partial T}{\partial \eta}.$$

(b) Express q_x and q_y in terms of $\partial T/\partial x$ and $\partial T/\partial y$, and rearrange the result according to the geometric considerations so that

$$q_x = -(k_\xi \sin^2\alpha + k_\eta \cos^2\alpha)\frac{\partial T}{\partial x} - (k_\xi - k_\eta)\sin\alpha\cos\alpha\frac{\partial T}{\partial y},$$

$$q_y = -(k_\xi - k_\eta)\cos\alpha\sin\alpha\frac{\partial T}{\partial x} - (k_\xi \cos^2\alpha + k_\eta \sin^2\alpha)\frac{\partial T}{\partial y}.$$

(c) Show that the two-dimensional conduction equation for homogeneous anisotropic materials is

$$\rho c \frac{\partial T}{\partial t} = (k_\xi \sin^2\alpha + k_\eta \cos^2\alpha)\frac{\partial^2 T}{\partial x^2} + (k_\xi \cos^2\alpha + k_\eta \sin^2\alpha)\frac{\partial^2 T}{\partial y^2} + (k_\xi - k_\eta)\cos 2\alpha \frac{\partial^2 T}{\partial x\,\partial y}.$$

(d) What is the special form of the foregoing equation corresponding to homogeneous isotropic materials? (e) What value for the conductivity would be obtained if the plate were held between the isothermal surfaces of a conductivity test apparatus? (f) Show that under the conditions of part (e),

$$\tan\beta = \frac{q_\eta}{q_\xi} = \frac{k_\eta(\partial T/\partial \eta)}{k_\xi(\partial T/\partial \xi)} = \frac{k_\eta[\sin\alpha(\partial T/\partial y) - \cos\alpha(\partial T/\partial x)]}{k_\xi[\cos\alpha(\partial T/\partial y) + \sin\alpha(\partial T/\partial x)]};$$

show also that given $\partial T/\partial x = 0$,

$$\tan\beta = \frac{k_\eta}{k_\xi}\tan\alpha,$$

which implies $\beta < \alpha$, and that the heat flux vector is no longer normal to isothermal surfaces as it would be if the material were isotropic.

2–19. A frictionless incompressible fluid having the velocity U flows steadily through a tube (Fig. 2–58). The inlet temperature of the fluid is T_0. The initial temperatures of the tube and fluid are the same and equal to the inlet temperature of the fluid. The uniform heat flux q'' is suddenly applied to the outer surface of the tube.

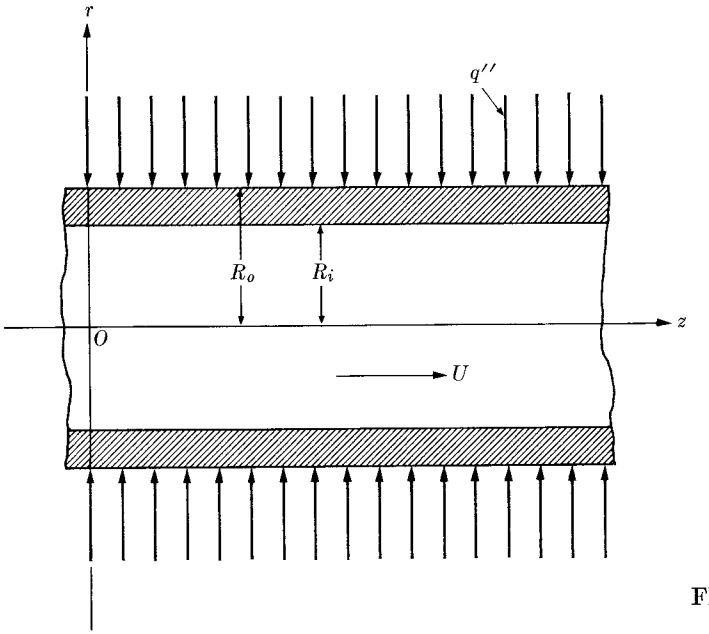


FIG. 2–58

The axial conduction is negligible. Give the differential formulation of the problem, employing only one of the following assumptions at a time: (i) None. (ii) The thickness of the tube is negligible. (iii) The thickness of the tube is small. (iv) The fluid is radially lumped, and the heat transfer coefficient is (a) large, or (b) moderate. (v) The fluid and tube are radially lumped.*

2–20. A frictionless incompressible fluid having the velocity U_∞ flows steadily over a semi-infinite plate of negligible thickness (Fig. 2–59). The temperatures of the fluid and the wall are T_∞ and T_w, respectively. Find the thickness of the steady penetration depth (boundary layer) of the temperature depending on x and the other variables involved.

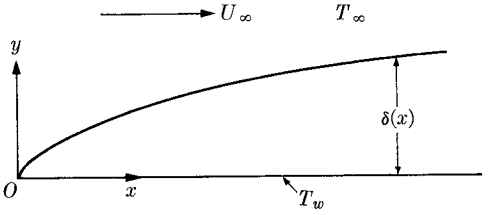


FIG. 2–59

* Note that a formulation based on assumption (i), although it is more exact, is difficult to solve. Formulations based on (ii) and (iii), without any or with only partial effect of the tube wall, are models for fluid flow problems. On the other hand, formulations using (iv-a) and (iv-b), allowing radial variation in the tube but not in the fluid, become convenient for thermal stress problems. The formulation based on (v), being completely lumped in the radial direction, is the simplest model. This may be utilized to evaluate the bulk temperature of the fluid at a desired location.

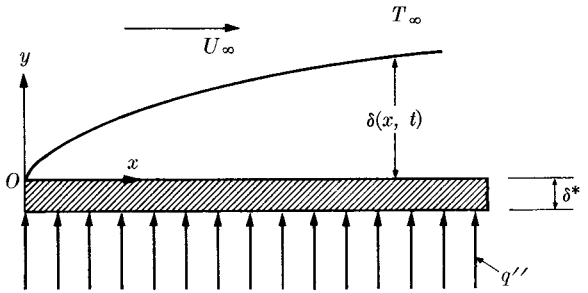
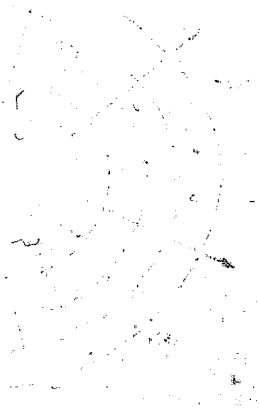


FIG. 2–60

2–21. A frictionless incompressible fluid having the velocity U_∞ flows steadily over a thin semi-infinite plate of thickness δ^* (Fig. 2–60). The initial temperatures of the fluid and the wall are the same, T_∞. Then the uniform heat flux q'' is suddenly applied to the bottom of the plate. Find the unsteady penetration of the heat flux into the fluid.

PART II | SOLUTION





2-1. A General Problem

STEADY ONE-DIMENSIONAL PROBLEMS. BESSEL FUNCTIONS

The lumped and integral formulations of steady one-dimensional problems involve the solving of algebraic equations only. The differential formulation of the same problems, on the other hand, involves the solving of first- or second-order differential equations. As we mentioned in Chapter 2, methods for solving linear differential equations of first order, and of second order with constant coefficients, are assumed to be known to the reader. Hence for steady one-dimensional problems resulting in such equations we shall give the solution. For problems of temperature-dependent thermal conductivity the differential formulation yields a nonlinear differential equation of the second order, and for problems of space-dependent thermal conductivity, as well as for those of extended surfaces with variable cross sections, the differential formulation results in a linear differential equation of second order with variable coefficients. In Sections 3–5 and 3–6, we shall introduce the methods appropriate to solution of both these types of equations. First, however, we shall consider a general problem in which a number of important concepts are defined.

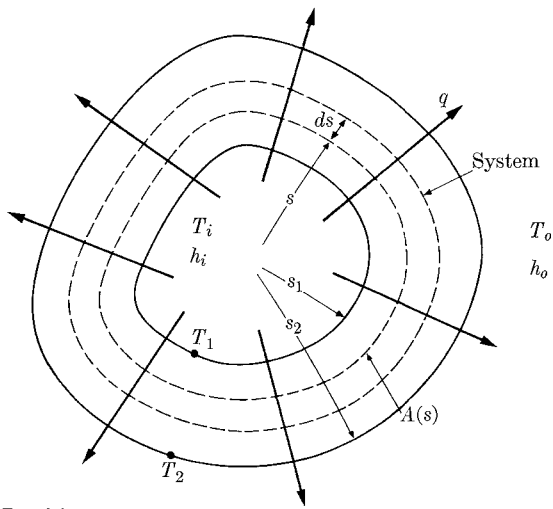


FIG. 3–1

3–1. A General Problem

Consider a long, hollow cylinder or a thick-walled, closed shell of constant wall thickness whose cross section is shown in Fig. 3–1. This cylinder or shell contains a fluid at temperature T_i and is surrounded by an ambient at temperature T_o. Let us suppose that $T_i > T_o$. The inside and outside heat transfer coefficients are h_i and h_o, respectively. We wish to know the temperature distribution of and the heat transfer through this cylinder or shell.

The method of solution employed here is convenient for one-dimensional problems in which q = const at every cross section. The first two steps of the formulation (see Section 2–9) result in

$$d[q_s A(s)] = 0, \tag{3–1}$$

which readily gives

$$q = q_s A(s) = \text{const}, \tag{3–2}$$

where s denotes the space variable and $A(s)$ the corresponding heat transfer area. The third step of the formulation is

$$q_s = -k \frac{dT}{ds}. \tag{3–3}$$

The fourth step is the introduction of Eq. (3–3) into Eq. (3–2). Rearranging and integrating the result between the inner and outer surfaces with the assumption that k is constant, we get

$$q = \frac{T_1 - T_2}{(1/k)\int_{s_1}^{s_2} ds/A(s)} = \frac{T_1 - T_2}{R}, \tag{3–4}$$

where

$$R = \frac{1}{k}\int_{s_1}^{s_2} \frac{ds}{A(s)} \tag{3–5}$$

is the so-called *conductive resistance*, and the subscripts 1 and 2 refer to the inner and outer surfaces, respectively. Equation (3–4) is analogous to Ohm's law,

$$i = \frac{E_1 - E_2}{R_e},$$

for steady electric current; the conduction heat transfer q corresponds to the electric current i, the temperature drop $T_1 - T_2$ to the potential drop $E_1 - E_2$, and the conductive resistance R to the electrical resistance R_e. (What is the differential form of Ohm's law?)

Since ambient temperatures are more easily measured than surface temperatures, it is convenient to express q in terms of the ambient temperature. From the last step of the formulation, we have

$$q = \frac{T_i - T_1}{1/h_i A(s_1)} = \frac{T_i - T_1}{R_i}, \tag{3–6}$$

where

$$R_i = \frac{1}{h_i A(s_1)} \tag{3–7}$$

is the *convective resistance* between the inside fluid and the inner surface of the wall; similarly, we have

$$q = \frac{T_2 - T_o}{1/h_o A(s_2)} = \frac{T_2 - T_o}{R_o}, \tag{3–8}$$

where

$$R_o = \frac{1}{h_o A(s_2)} \tag{3-9}$$

is the outside convective resistance. Solving Eqs. (3–4), (3–6), and (3–8) for their respective temperature differences, then adding the results side by side eliminates T_1 and T_2, and yields

$$q = \frac{T_i - T_o}{R_i + R + R_o}. \tag{3-10}$$

The same result may be readily obtained by considering the analogy between the diffusion of heat and electric current. Thus the problem becomes analogous to the evaluation of an electric current through three resistors connected in series (Fig. 3–2).

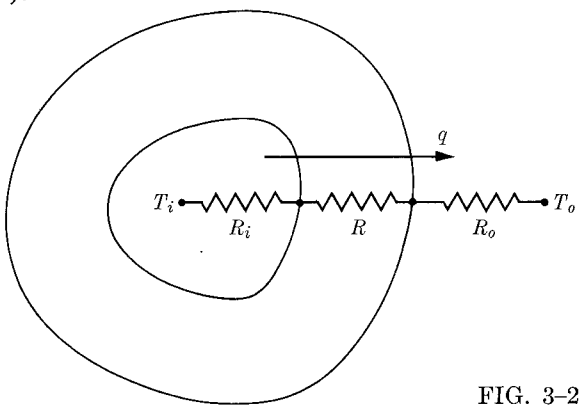


FIG. 3–2

It is sometimes convenient to simplify Eq. (3–10) by writing it in terms of the so-called *over-all coefficient of heat transfer* U, which is defined according to

$$q = UA(T_i - T_o), \tag{3-11}$$

and

$$\frac{1}{UA} = R_i + R + R_o. \tag{3-12}$$

Since U depends on A, the statement of U is ambiguous until an area is chosen. Noting that

$$UA = U_i A(s_1) = U_o A(s_2),$$

where U_i and U_o denote the over-all heat transfer coefficients based on the inner and outer surface areas, respectively, we may write the outside coefficient U_o, for example, as

$$\frac{1}{U_o} = \frac{A(s_2)/A(s_1)}{h_i} + \frac{A(s_2)}{k}\int_{s_1}^{s_2} \frac{ds}{A(s)} + \frac{1}{h_o}. \tag{3-13}$$

To determine the temperature distribution of the problem we reconsider Eq. (3–4) now integrated from an arbitrary location s to the outer surface* in the form

$$q = \frac{T - T_2}{(1/k)\int_s^{s_2} ds/A(s)}, \tag{3–14}$$

where T is the temperature of the location s. Elimination of T_2 between Eqs. (3–8) and (3–14) in the usual manner yields

$$q = \frac{T - T_o}{(1/k)\int_s^{s_2} ds/A(s) + 1/h_o A(s)}. \tag{3–15}$$

Finally, equating Eq. (3–11) to Eq. (3–13) and rearranging the result in terms of U_o, we obtain the desired temperature distribution:

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{A(s_2)}{k}\int_s^{s_2} \frac{ds}{A(s)} + \frac{1}{h_o}\right]. \tag{3–16}$$

An alternative method of solution which leads first to the temperature distribution and then to the heat transfer may be obtained from the formal formulation of the problem. Thus the governing equation is found, from the first four steps of the formulation, to be

$$\frac{d}{ds}\left[A(s)\frac{dT}{ds}\right] = 0, \tag{3–17}$$

subject to the boundary conditions

$$\begin{aligned} +k\frac{dT(s_1)}{ds} &= h_i[T(s_1) - T_i], \\ -k\frac{dT(s_2)}{ds} &= h_o[T(s_2) - T_o], \end{aligned} \tag{3–18}$$

obtained from the last step of the formulation. Substituting the solution of Eq. (3–17),

$$T = C\int^s \frac{ds}{A(s)} + D, \tag{3–19}$$

into Eq. (3–18) results in

$$\begin{aligned} \frac{kC}{A(s_1)} &= h_i\left[C\int^{s_1} \frac{ds}{A(s)} + D - T_i\right], \\ -\frac{kC}{A(s_2)} &= h_o\left[C\int^{s_2} \frac{ds}{A(s)} + D - T_o\right]. \end{aligned} \tag{3–20}$$

*An alternative expression for the temperature distribution is obtained by carrying this integration out from the inner surface to the arbitrary location s.

The values of C and D from Eq. (3–20), introduced into Eq. (3–19), give the previously obtained temperature distribution, Eq. (3–16). The heat transfer may now be found by inserting Eq. (3–16) into the combination of Eqs. (3–2) and (3–3); this yields the previous result, Eq. (3–10).

For convenience, we shall apply the procedure of the foregoing problem to three important cases, the cartesian, cylindrical, and spherical geometries. The over-all heat transfer coefficients based on the outer surface area and the temperature distributions of these geometries are:

$$\frac{1}{U_o} = \frac{1}{U} = \frac{1}{h_i} + \frac{L}{k} + \frac{1}{h_o}, \qquad \text{cartesian,} \qquad (3\text{–}21)$$

$$\frac{1}{U_o} = \frac{(R_2/R_1)}{h_i} + \frac{R_2}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_o}, \qquad \text{cylindrical,} \qquad (3\text{–}22)$$

$$\frac{1}{U_o} = \frac{(R_2/R_1)^2}{h_i} + \frac{R_2}{k}\left(\frac{R_2}{R_1} - 1\right) + \frac{1}{h_o}, \qquad \text{spherical,} \qquad (3\text{–}23)$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left(\frac{x_2 - x}{L} + \frac{1}{h_o}\right), \qquad \text{cartesian,} \qquad (3\text{–}24)$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{R_2}{k}\ln\left(\frac{R_2}{r}\right) + \frac{1}{h_o}\right], \qquad \text{cylindrical,} \qquad (3\text{–}25)$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{R_2}{k}\left(\frac{R_2}{r} - 1\right) + \frac{1}{h_o}\right], \qquad \text{spherical.} \qquad (3\text{–}26)$$

3–2. Composite Structures

Assume that the hollow cylinder or the thick-walled, closed shell of Fig. 3–1 is composed of N layers of materials having different thicknesses and thermal conductivities (Fig. 3–3). The contact resistance between the layers is negligible. We wish to find the heat transfer from the inner fluid to the surrounding ambient, and the temperature distribution of the structure.

Extending the analogy between the diffusion of heat and electric current to the present case, we readily obtain

$$\frac{1}{UA} = R_i + \sum_{n=1}^{N} R_n + R_o. \qquad (3\text{–}27)$$

The explicit form of U based on the outside surface is

$$\frac{1}{U_o} = \frac{A(s_{N+1})/A(s_1)}{h_i} + A(s_{N+1})\sum_{n=1}^{N}\frac{1}{k_n}\int_{s_n}^{s_{n+1}}\frac{ds}{A(s)} + \frac{1}{h_o}. \qquad (3\text{–}28)$$

Equation (3–28) reduces to Eq. (3–13) for $N = 1$.

To obtain the temperature distribution in the structure we first express q in terms of the temperature difference $T - T_o$ and the corresponding resistances (from the series R_n, R_{n+1}, R_{n+2}, . . . , R_N). The result is

$$q = \frac{T - T_o}{(1/k_n)\int_s^{s_{n+1}} ds/A(s) + \sum_{m=n+1}^{N} (1/k_m)\int_{s_m}^{s_{m+1}} ds/A(s) + 1/h_o A(s_{N+1})}, \tag{3–29}$$

where T denotes the temperature of the location s (Fig. 3–3). Then eliminating q between Eqs. (3–11) and (3–29), we find that the desired temperature distribution in terms of U_o is

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{A(s_{N+1})}{k_n}\int_s^{s_{n+1}} \frac{ds}{A(s)} + A(s_{N+1})\sum_{m=n+1}^{N} \frac{1}{k_m}\int_{s_m}^{s_{m+1}} \frac{ds}{A(s)} + \frac{1}{h_o}\right]. \tag{3–30}$$

Equation (3–30) reduces to Eq. (3–16) for $N = 1$.

The cartesian, cylindrical, and spherical forms of Eqs. (3–28) and (3–30) are listed below:

$$\frac{1}{U_o} = \frac{1}{U} = \frac{1}{h_i} + \sum_{n=1}^{N} \frac{L_n}{k_n} + \frac{1}{h_o}, \qquad \text{cartesian,} \tag{3–31}$$

$$\frac{1}{U_o} = \frac{(R_{N+1}/R_1)}{h_i} + R_{N+1}\sum_{n=1}^{N} \frac{1}{k_n}\ln\left(\frac{R_{n+1}}{R_n}\right) + \frac{1}{h_o}, \qquad \text{cylindrical,} \tag{3–32}$$

$$\frac{1}{U_o} = \frac{(R_{N+1}/R_1)^2}{h_i} + R_{N+1}^2\sum_{n=1}^{N} \frac{1}{k_n}\left(\frac{1}{R_n} - \frac{1}{R_{n+1}}\right) + \frac{1}{h_o}, \qquad \text{spherical,} \tag{3–33}$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{x_{n+1} - x}{k_n} + \sum_{m=n+1}^{N} \frac{x_{m+1} - x_m}{k_m} + \frac{1}{h_o}\right], \qquad \text{cartesian,} \tag{3–34}$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{R_{N+1}}{k_n}\ln\left(\frac{R_{n+1}}{r}\right) + R_{N+1}\sum_{m=n+1}^{N} \frac{1}{k_m}\ln\left(\frac{R_{m+1}}{R_m}\right) + \frac{1}{h_o}\right], \qquad \text{cylindrical,} \tag{3–35}$$

$$\frac{T - T_o}{T_i - T_o} = U_o\left[\frac{R_{N+1}}{k_n}\left(\frac{R_{N+1}}{r} - \frac{R_{N+1}}{R_{n+1}}\right) + R_{N+1}^2\sum_{m=n+1}^{N} \frac{1}{k_m}\left(\frac{1}{R_m} - \frac{1}{R_{m+1}}\right) + \frac{1}{h_o}\right], \qquad \text{spherical.} \tag{3–36}$$

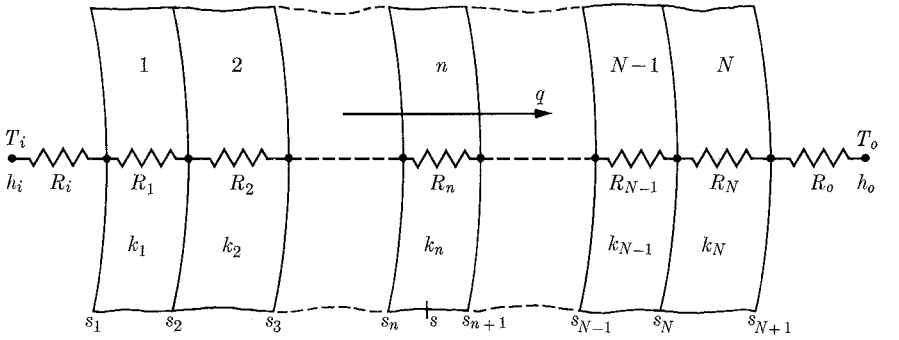


FIG. 3–3

In practice, the combination of series- and parallel-connected structures is also important, especially in cartesian geometry. For example, consider a wall composed of hollow concrete blocks, such as is used in building construction (Fig. 3–4). Actually, the heat transfer through this type of wall is not one-dimensional. However, a one-dimensional analysis gives satisfactory results for practical problems.

Again employing the electrical analogy, we readily obtain

$$\frac{1}{UA} = R_i + R_1 + \frac{1}{1/R_3 + 1/R_4 + 1/R_5} + R_2 + R_o.$$

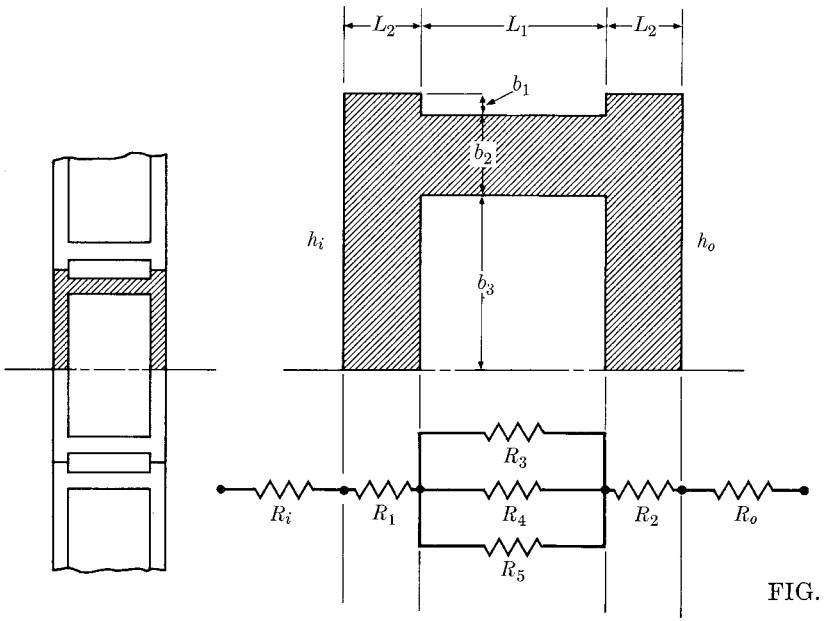


FIG. 3–4

Thus we have, per unit width of the wall perpendicular to the cross section shown in Fig. 3–4,

$$\frac{1}{h_i(b_1 + b_2 + b_3)} + \frac{L_2}{k_2(b_1 + b_2 + b_3)} + \frac{1}{k_1 b_1/L_1 + k_2 b_2/L_1 + k_1 b_3/L_1} + \frac{L_2}{k_2(b_1 + b_2 + b_3)} + \frac{1}{h_o(b_1 + b_2 + b_3)},$$

and hence per unit area of the wall

$$\frac{1}{U} = \frac{1}{h_i} + \frac{L_2}{k_2} + \frac{L_1}{\epsilon_1 k_1 + \epsilon_2 k_2} + \frac{L_2}{k_2} + \frac{1}{h_o},$$

where

$$\epsilon_1 = (b_1 + b_3)/(b_1 + b_2 + b_3) \quad \text{and} \quad \epsilon_2 = b_2/(b_1 + b_2 + b_3).$$

3–3. Examples

In this section a number of physical and mathematical facts are demonstrated in terms of examples selected from cartesian, cylindrical, and spherical geometries.

Example 3–1. A flat plate of thickness L separates two media having temperatures T_i and T_o. The heat transfer coefficients are h_i and h_o (Fig. 3–5). We wish to eliminate the heat loss from the ambient of higher temperature.

We learned in Section 3–2 that the total resistance between two ambients can be increased by adding insulation to one or both surfaces of the plate. Regardless of the thickness and material of the insulation, however, the heat loss from the ambient of higher temperature cannot be completely eliminated

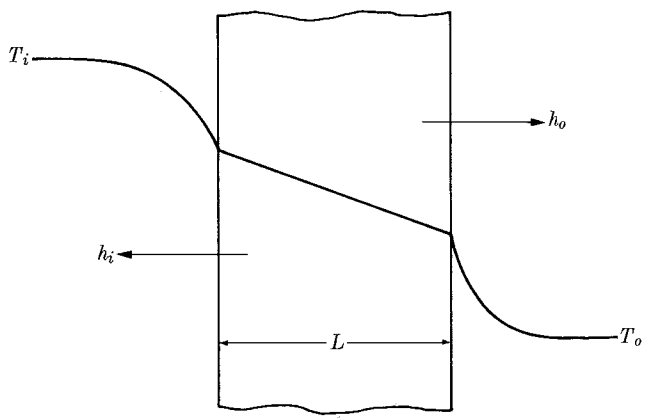


FIG. 3–5

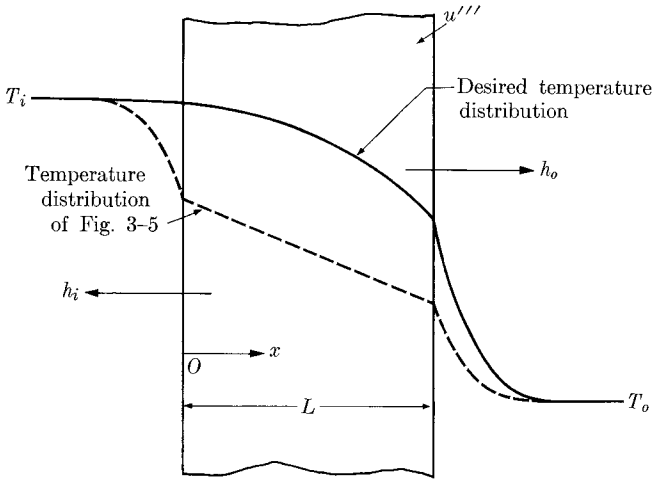


FIG. 3–6

by such insulation. On the other hand, a proper amount of uniform internal energy u''' generated electrically in the plate* may reduce the heat loss to zero (Fig. 3–6). The problem is thus reduced to finding the appropriate value of u'''.

The formulation of the problem in x measured from the left surface of the plate rightward is

$$\frac{d^2T}{dx^2} + \frac{u'''}{k} = 0, \tag{3–37}$$

$$T(0) = T_i, \tag{3–38}$$

$$\frac{dT(0)}{dx} = 0, \tag{3–39}$$

$$-k\frac{dT(L)}{dx} = h_o[T(L) - T_o]. \tag{3–40}$$

[Why three boundary conditions rather than two? What are the unknowns to be determined? Describe the physics of the problem as given by Eqs. (3–37), (3–38), (3–40) and as given by Eqs. (3–37), (3–39), (3–40).]

Introducing the solution of Eq. (3–37),

$$T = -\frac{u'''x^2}{2k} + Ax + B, \tag{3–41}$$

* If the plate is not an electrical conductor or if electric current through the plate is undesirable, another electrically heated plate (guard heater), electrically insulated from but in good thermal contact with the original plate, can be utilized. See also the problem related to Fig. 2–16.

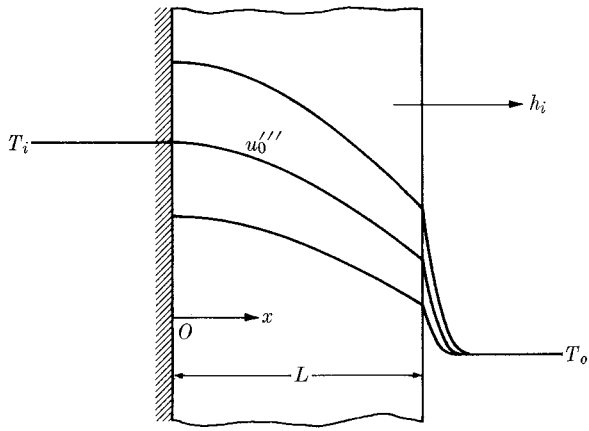


FIG. 3–7

into Eqs. (3–39) and (3–40), we obtain

$$0 = A, \qquad u'''L = h_o(-u'''L^2/2k + B - T_o).$$

Thus, by inserting the values of A and B into Eq. (3–41), we find that the temperature distribution corresponding to an arbitrary value of u''' is

$$\frac{T - T_o}{u'''L^2/2k} = 1 - \left(\frac{x}{L}\right)^2 + 2\left(\frac{k}{h_oL}\right). \tag{3–42}$$

A sketch of Eq. (3–42) is given in Fig. 3–7 for various values of u'''. Among these temperature profiles, however, there is only one which satisfies the remaining boundary condition, Eq. (3–38), and which therefore corresponds to the desired value u_o''' of u'''. Thus, combining Eqs. (3–38) and (3–42), we have

$$\frac{u_o'''L^2}{2k} = \frac{T_i - T_o}{1 + 2(k/h_oL)}.$$

[Re-solve this example by considering Eq. (3–39) or Eq. (3–40) as the last boundary condition to be used.]

Example 3–2. The fuel element of a pool reactor is composed of flat plates of thickness $2L_1$ and cladding material of thickness $(L_2 - L_1)$ bonded to the surfaces of these plates (Fig. 3–8). Uniform (nuclear) internal energy u''' is assumed to be generated in the plates only. The heat transfer coefficient is h, the temperature of the coolant T_∞. We need to know the temperature distribution of the fuel element.

This is an example of a *multidomain* problem. The formulation of such problems involves more than one governing equation. Because of the geometric and thermal symmetries, x is measured from the middle plane of the fuel element. Denoting properties of the plate and of the cladding by the subscripts 1

and 2, respectively, we obtain

$$\frac{d^2T_1}{dx^2} + \frac{u'''}{k_1} = 0, \qquad 0 \leq x \leq L_1, \tag{3–43}$$

$$\frac{d^2T_2}{dx^2} = 0, \qquad L_1 \leq x \leq L_2, \tag{3–44}$$

$$\frac{dT_1(0)}{dx} = 0,$$

$$T_1(L_1) = T_2(L_1) = T_{12}, \qquad k_1 \frac{dT_1(L_1)}{dx} = k_2 \frac{dT_2(L_1)}{dx}, \tag{3–45}$$

$$-k_2 \frac{dT_2(L_2)}{dx} = h[T_2(L_2) - T_\infty].$$

Introducing the solution of Eq. (3–43),

$$T_1 = -\frac{u'''x^2}{2k_1} + Ax + B, \tag{3–46}$$

and that of Eq. (3–44),

$$T_2 = Cx + D, \tag{3–47}$$

into the boundary conditions, Eq. (3–45), we obtain the following four simultaneous algebraic equations:

$$A = 0, \qquad -u'''L_1^2/2k_1 + B = CL_1 + D,$$
$$-u'''L_1 = k_2C, \qquad -k_2C = h(CL_2 + D - T_\infty).$$

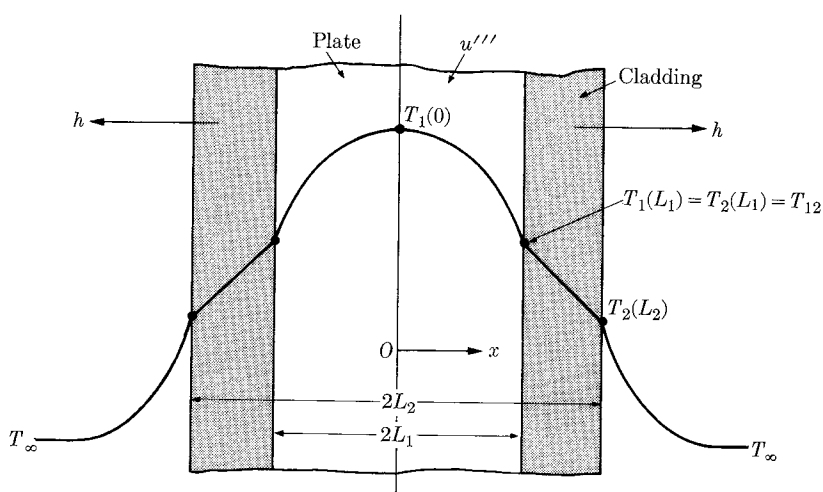


FIG. 3–8

Solving these in terms of A, B, C, and D, then inserting the result into Eqs. (3–46) and (3–47) gives the temperature distribution of the fuel element as

$$\frac{T_1 - T_\infty}{u'''L_1^2/2k_1} = 1 - \left(\frac{x}{L_1}\right)^2 - 2\left(\frac{k_1}{k_2}\right) - 2\left(\frac{L_2}{L_1}\right)\left(\frac{k_1}{k_2}\right)\left(1 + \frac{k_2}{hL_2}\right) \tag{3–48}$$

for $0 \leq x \leq L_1$, and

$$\frac{T_2 - T_\infty}{u'''L_1^2/k_2} = -\left(\frac{x}{L}\right) + \left(\frac{L_2}{L_1}\right)\left(1 + \frac{k_2}{hL_2}\right) \tag{3–49}$$

for $L_1 \leq x \leq L_2$.

When we wish to know only the temperature at some specific locations of the fuel element, say $T_1(0)$, $T_1(L_1) = T_2(L_1) = T_{12}$, and $T_2(L_2)$, the electrical analogy may be employed in place of the foregoing procedure. Thus we have

$$T_2(L_2) - T_\infty = \frac{1}{h}(u'''L_1),$$

$$T_{12} - T_\infty = \left(\frac{1}{h} + \frac{L_2 - L_1}{k}\right)(u'''L_1),$$

where $(u'''L_1)$ is the heat flux across each surface of the fuel element. Furthermore, noting from Eq. (3–27) that

$$T_1(0) - T_{12} = -\frac{u'''L_1^2}{2k_1},$$

we obtain

$$T_1(0) - T_\infty = \left(\frac{L_1}{2k_1} + \frac{1}{h} + \frac{L_2 - L_1}{k_2}\right)(u'''L_1).$$

Example 3–3. A constant-property inviscous liquid having the far upstream temperature T_0 and velocity V flows steadily through an infinitely long tube of cross-sectional area A and periphery P. The wall thickness of the tube is negligible. The downstream half of the tube is subjected to the constant heat flux q''; the upstream half is insulated (Fig. 3–9). We wish to know the axial temperature distribution of the liquid, based on a radially lumped analysis.

This problem will be investigated in detail because it finds considerable application in reactor technology and the computation of heat transfer coefficients in tubes. Let us first clarify an important fact which is not obvious from the statement of the problem. The applied heat flux q'' is going to be axially conducted through the fluid from the heated to the unheated half of the tube, thus giving rise to a temperature distribution in the unheated half. Since the axial enthalpy flow is in the direction opposite to that of the conduction, it has a reverse effect on the temperature rise. Thus the temperature distribution

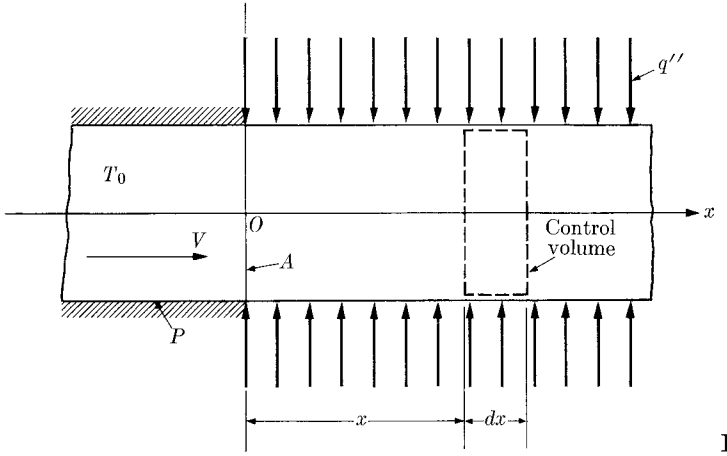


FIG. 3–9

in the tube depends on the relative importance of axial conduction and axial enthalpy flow. Although the fluid constitutes a single domain, the change in the peripheral boundary condition from insulation to constant heat flux suggests that we treat this as a two-domain problem for mathematical convenience. The formulation of the unheated section may be readily obtained by substituting $q'' = 0$ into that of the heated section; therefore, details of the formulation are given only for the heated section.

Consider the radially lumped, axially differential control volume shown in Fig. 3–9. Let us apply the general laws to this control volume as follows.

Conservation of mass:

$$V = \text{const.} \tag{3–50}$$

Momentum (mechanical energy or Bernoulli's equation):*

$$\frac{d}{dx}\left(\frac{p}{\rho} + gz\right) = 0. \tag{3–51}$$

First law of thermodynamics (total energy), in terms of Fig. 3–10:

$$-A\frac{dq_x}{dx} - \rho AV\frac{dh_0}{dx} + q''P = 0. \tag{3–52}$$

It follows from the definition of h_0, Eqs. (3–50) and (3–51), that

$$dh_0/dx = du/dx. \tag{3–53}$$

Neglecting $c_p - c_v$ for liquids, we have, in terms of a common specific heat c,

$$du = c\,dT. \tag{3–54}$$

* See the development of Eq. (2–56).

Introducing Eq. (3–54) into Eq. (3–53) and the result into Eq. (3–52), we get for the thermal energy

$$-A\frac{dq_x}{dx} - \rho cAV\frac{dT}{dx} + q''P = 0. \tag{3–55}$$

Finally, Eq. (3–55), rearranged with the x-component of Fourier's law,

$$q_x = -k\frac{dT}{dx},$$

gives the governing equation of the heated section in the following form:

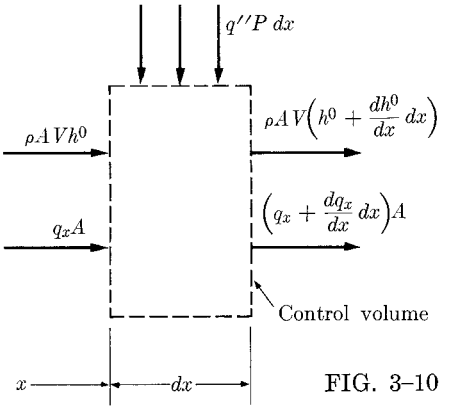


FIG. 3–10

$$\frac{d^2T}{dx^2} - \frac{\rho cV}{k}\frac{dT}{dx} + \frac{q''P}{kA} = 0. \tag{3–56}$$

[Modify Eq. (3–56) assuming that the tube of Fig. 3–9 is vertical. Reformulate the problem for an ideal gas.] With $q'' = 0$, Eq. (3–56) becomes the governing equation of the unheated section.

Distinguishing the temperatures of the unheated and heated sections by the subscripts 1 and 2, respectively, we thus have the formulation of the problem as follows:

$$\begin{aligned} &\frac{d^2T_1}{dx^2} - \frac{\rho cV}{k}\frac{dT_1}{dx} = 0, \qquad -\infty < x \leq 0, \\ &\frac{d^2T_2}{dx^2} - \frac{\rho cV}{k}\frac{dT_2}{dx} + \frac{q''P}{kA} = 0, \qquad 0 \leq x < +\infty, \end{aligned} \tag{3–57}$$

subject to

$$\begin{aligned} &T_1(-\infty) = T_0, \\ &T_1(0) = T_2(0), \qquad \frac{dT_1(0)}{dx} = \frac{dT_2(0)}{dx}, \\ &T_2(\infty) \sim x, \end{aligned} \tag{3–58}$$

where the last boundary condition indicates a linear temperature distribution in the liquid for large values of x. (If this point is not clear to the reader, he should first consider the general solution of the heated section, then ask himself whether an exponential temperature increase in this section is possible for a constant q''.)